EULERIAN FORMULATION FOR LARGE-DISPLACEMENT ANALYSIS OF SPACE FRAMES

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ABSTRACT: In this paper, a new general procedure is presented for modeling the effects of large displacements on the response of space frames subjected to conservative loading. An incremental definition of rotations is adopted based on an improved rotational transformation matrix, and a convected (Eulerian) system is employed for establishing the contribution of individual elements to the strain energy (U). The Eulerian displacements are obtained by means of element-based local vectors in which the vectors representing the principal axes of bending follow the deformed configuration of the element and are continuously updated to a position normal to the element chord. The nonlinear solution procedure is formalized in terms of transformations between the Eulerian and the global systems, and expressions for geometric stiffness and transformation matrices are explicitly derived. Verification examples utilizing an elastic quartic formulation and employing the nonlinear analysis program ADAPTIC are presented to demonstrate the accuracy and versatility of this method in the large-displacement analysis of space frames.

INTRODUCTION

The structural design process has evolved over the last few decades to allow safer, yet more economic, structures. This has largely been due to an increased understanding of the fundamental principles governing the structural behavior up to and beyond the ultimate limit state, including such effects as material and geometric nonlinearities. In the context of space frames, the study of geometric nonlinearity effects has always been complicated by the fact that the effect of finite rotations about fixed axes in three-dimensional space is sequence-dependent, and that moments corresponding to such rotations in expressions for work are not fixed in direction (Argyris et al. 1978a, b).

The advent of powerful computers has prompted a surge of research activities aiming at establishing methods that, although numerically demanding, are capable of accurately modeling the large-displacement behavior of structures. Of these, the displacement-based finite-element method has been most widely applied, mainly because of its accuracy, fully established mathematical basis, and suitability for computer implementation. The finite-element method is based on the general principle that structural equilibrium under applied loading is achieved at displacements that correspond to stationary total potential energy of the structure, as expressed by

Equilibrium \( \frac{\partial \prod}{\partial U_j} = 0 (\forall j) \) ................................................................ (1a)
\[ \prod = U + W \] .......................... (1b)

where \( U_j \) = nodal displacement at freedom \((j)\); \( U \) = strain energy; and \( W \) = loss of potential energy of the load.

At each node, translational and rotational freedoms are needed to define possible deformation shapes of the structure. Since the effect of finite rotations is sequence-dependent, the order of application of the rotations must be assumed (Besseling 1977). Alternatively, commutative definitions, such as semitangential rotations (Argyris et al. 1978a, b), or incremental application of rotations (Oran 1973) may be employed.

The strain energy \((U)\) can be obtained once the structure deformation shape is established in terms of nodal freedoms. On the other hand, the expression for the loss of potential energy of the load \((W)\) is sensitive to the type of applied moments and the particular definition of rotations employed. For example, moments corresponding to an incremental application of rotations are almost moments about fixed axes; and moments corresponding to sequential finite rotations are neither fixed in direction nor orthogonal. However, since realistic structural loading rarely includes applied moments, such considerations are only of academic interest.

Here, a new general procedure is proposed for modeling the effects of large displacements on the response of space frames to conservative loading. The procedure utilizes a convected (Eulerian) system for the determination of the element strain energy and models the effect of large rotations incrementally. Verification analyses demonstrate the potential of the developed method in the accurate and efficient nonlinear analysis of space frames.

**TECHNIQUE FORMALIZATION**

In modeling realistic space frames, the finite-element method requires a number of elements to be used; the deformation within each element is obtained from displacements at nodal points. Apart from the fact that most realistic loading does not include applied moments, the inclusion of moment loads results in complexities in the expression for the loss of potential energy of the load \((W)\), rendering it dependent on the type of moments and the particular definition of rotations. Assuming that the loading consists only of forces applied at the nodal positions, the total potential energy can be expressed as

\[ \prod = \sum_e (\varepsilon U) - \sum_j \mathbf{P}_j \mathbf{U}_j \] .......................... (2)

where \( \varepsilon U \) = element strain energy; \( \mathbf{P}_j \) = global applied force at freedom \((j)\); and \( \mathbf{U}_j \) = global nodal displacement of freedom \((j)\).

At equilibrium, \( \Pi \) is always stationary with respect to the nodal freedoms, hence

\[ \frac{\partial \prod}{\partial \mathbf{U}_j} = 0 \Rightarrow \sum_e \frac{\partial (\varepsilon U)}{\partial \mathbf{U}_j} - \mathbf{P}_j = 0 \Rightarrow \] .......................... (3)

\[ \mathbf{P}_j = \sum_e (\varepsilon \mathbf{f}_j) \] .......................... (4)

where

\[ \varepsilon \mathbf{f}_j = \frac{\partial (\varepsilon U)}{\partial \mathbf{U}_j} \] .......................... (5)
According to (4), stationary total potential energy of the structure is achieved if the global applied forces $P$ are identical to the resistance forces assembled from element contributions $f$. Although the loading vector $P$ does not include any moment loads, terms of $f$ corresponding to rotational freedoms are not necessarily zero. As discussed later, these terms correspond to element global moments whose types depend on the adopted definition of rotations. However, such consideration does not affect the finite-element solution, since the total potential energy is stationary relative to all possible nodal displacements and orientations.

**Nature of Global Rotations**

An incremental approach is adopted for the application of global rotations, an essential requirement for the development of a plastic hinge formulation (Izzuddin 1991). Oran (1973) used a first-order rotation matrix to describe the transformation of vector $a$ due to an increment of global rotations ($\alpha$, $\beta$, and $\gamma$) about the three global axes ($X$, $Y$, and $Z$)

$$a' = \sum_{j=1}^{3} r_{ij}a_j$$

with

$$r_T = \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix}$$

However, this relationship applies to very small increments, since the orthogonality property of transformed unit vectors is satisfied only to the first order in rotations.

In the current work, a higher-order transformation is employed. The derivation is obtained by applying the resultant rotation increment $\mid r \mid$ about the resultant rotation axis $r$, as shown in Fig. 1. The ensuing transformation is expressed as

![Fig. 1. Application of Global Rotations](image)
\[ a_i' = \sum_{j=1}^{3} T_{i,j} a_j \] ........................................ (7)

where

\[ T = \begin{bmatrix}
1 - (\beta^2 + \gamma^2)/2 & -\gamma + \alpha\beta/2 & \beta + \alpha\gamma/2 \\
\gamma + \alpha\beta/2 & 1 - (\alpha^2 + \gamma^2)/2 & -\alpha + \beta\gamma/2 \\
-\beta + \alpha\gamma/2 & \alpha + \beta\gamma/2 & 1 - (\alpha^2 + \beta^2)/2
\end{bmatrix} \] ......... (8)

The use of such a transformation reduces the amount of spurious lengthening of vectors upon incremental rotation. It also preserves to a greater extent the orthogonality property of unit vectors, as may be shown by the product of the various rows and columns of \( T \) in (8).

**NATURE OF GLOBAL MOMENTS**

As discussed in the earlier section entitled "Technique Formalization," the loading vector \( \mathbf{P} \) is assumed not to include any applied moments, but the global element resistance vector \( \mathbf{a}' \) may contain nonzero moment terms. Although the nature of these moments does not affect the finite-element solution inasmuch as displacements are concerned, the investigation of the effect of the definition of rotation adopted herein on these moments is found to be extremely instructive.

In the virtual-work expression, the moments corresponding to the global rotation increments of the previous section are not fixed in direction. This can be demonstrated by equating the virtual work of these moments to that of moments about fixed axes, i.e.:

\[ \sum_{i=1}^{3} m_i \delta r_i = \sum_{j=1}^{3} m_j \delta r_j \] ........................................ (9)

where, \( \delta r = \) infinitesimal increment of rotations about fixed axes; and \( \delta r = \) infinitesimal increment of rotations as defined in the section headed "Nature of Global Rotations."

The relationship between \( \delta r \) and \( \delta r' \) can be obtained by establishing the equivalence of their effects on the rotated vector \( \mathbf{a}' \) as follows:

\[ \delta a_i' = \sum_{j=1}^{3} \delta_i T_{i,j} a_j = \sum_{k=1}^{3} \delta_i' T_{i,k} a_k' \] ........................ (10)

\[ \sum_{j=1}^{3} \delta_i T_{i,j} a_j = \sum_{k=1}^{3} \delta_i' T_{i,k} \sum_{j=1}^{3} T_{k,j} a_j = \sum_{j=1}^{3} \sum_{k=1}^{3} \delta_i' T_{i,k} r T_{k,j} a_j \] .......................... (11)

\[ \delta_i T_{i,j} = \sum_{k=1}^{3} \delta_i' T_{i,k} r T_{k,j} \] ........................................ (12)

These relationships represent a set of nine equations with only three unknowns (\( \delta x, \delta \beta, \) and \( \delta \gamma \)). Thus, a unique solution exists only for a transformation matrix \( \mathbf{T} \) preserving exactly the orthogonality property of unit vectors. Since \( \mathbf{T} \) in (8) is accurate to the second-order in rotation, the solution of (10) can only be obtained to the first order.

From (8) it can be established that
\[ \delta T = \begin{bmatrix}
-\beta \delta \beta - \gamma \delta \gamma & -\delta \gamma & \frac{\alpha \delta \beta + \beta \delta \alpha}{2} & \delta \beta + \frac{\alpha \delta \gamma + \gamma \delta \alpha}{2}
\end{bmatrix} \]

and to the first order

\[ \delta T = \begin{bmatrix}
1 & -\gamma & \beta
\end{bmatrix} \begin{bmatrix}
\gamma & 1 & -\alpha
\end{bmatrix} \]

and the infinitesimal transformation due to rotations about fixed axes is given by

\[ \delta' T = \begin{bmatrix}
0 & -\delta \gamma' & \delta \beta'
\end{bmatrix} \begin{bmatrix}
\delta \gamma & 0 & -\delta \alpha
\end{bmatrix} \begin{bmatrix}
-\delta \beta' & \delta \alpha' & 0
\end{bmatrix} \]

Using (13)-(15), it can be demonstrated that the first-order solution to (12)

\[ \begin{bmatrix}
\delta \alpha
\end{bmatrix} = \begin{bmatrix}
1 & \gamma/2 & -\beta/2
\end{bmatrix} \begin{bmatrix}
\delta \alpha'
\end{bmatrix} \]

\[ \begin{bmatrix}
\delta \beta
\end{bmatrix} = \begin{bmatrix}
-\gamma/2 & 1 & \alpha/2
\end{bmatrix} \begin{bmatrix}
\delta \beta'
\end{bmatrix} \]

\[ \begin{bmatrix}
\delta \gamma
\end{bmatrix} = \begin{bmatrix}
\beta/2 & -\alpha/2 & 1
\end{bmatrix} \begin{bmatrix}
\delta \gamma'
\end{bmatrix} \]

or

\[ \delta r_i = \sum_{i=1}^{3} m T_{i,j} \delta r_i \]

where

\[ m T = \begin{bmatrix}
1 & -\gamma/2 & \beta/2
\end{bmatrix} \begin{bmatrix}
\gamma/2 & 1 & -\alpha/2
\end{bmatrix} \begin{bmatrix}
-\beta/2 & \alpha/2 & 1
\end{bmatrix} \]

The combination of (9) and (18) demonstrates that the global moments corresponding to the current definition of rotations are related to moments about fixed axes by the expression

\[ f m_i = \sum_{j=1}^{3} m T_{i,j} m_j \]

It is therefore evident that the current global moments are not moments about fixed axes but are, rather, defined in a system rotated by half the increment of rotations (\( \alpha \), \( \beta \), and \( \gamma \)), as shown in Fig. 2. This corresponds to the semitangential moment definition (Argyris et al. 1978), only it is applied in the context of an incremental approach. However, as long as the increments of rotation (\( \alpha \), \( \beta \), and \( \gamma \)) are small, \( m T \) is approximately an identity matrix, and the global moments can be assumed to be about fixed axes.
FIG. 2. Orientation of Global Moments Relative to Fixed Global Axes

FIG. 3. Eulerian (Convected) System in Relation to Global Reference System

EULERIAN SYSTEM

The Eulerian (convected) system is a local reference system that follows the element chord during deformation, as shown in Fig. 3. This system is convenient for the definition of the element strain energy ($^e U$) in the pres-
ence of large nodal displacements, since it isolates the components of displace-
ment resulting in element strains from displacements associated with stress-
free rigid-body motion.

We first define vector \( \mathbf{g}\mathbf{u} \) as the incremental global element displace-
ments (Fig. 3), and vector \( \mathbf{c}\mathbf{u} \) as the total local element displacements (Fig. 4); i.e.

\[
\mathbf{g}\mathbf{u} = (u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2)^T \quad \cdots \quad (20)
\]

\[
\mathbf{c}\mathbf{u} = (\theta_{1y}, \theta_{1z}, \theta_{2y}, \theta_{2z}, \Delta, \theta_T)^T \quad \cdots \quad (21)
\]

To obtain the local Eulerian displacements \( \mathbf{c}\mathbf{u} \), the effect of the incremen-
tal global displacements and rotations \( \mathbf{g}\mathbf{u} \) on a set of chosen element vectors is
quantified. These vectors represent the element chord and the two principal
axes of bending at the two ends, as shown in Fig. 5. This approach is different
from that employed in previous investigations (Oran 1973; Meek and Lo-

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**FIG. 4. Local Eulerian Displacements**

**FIG. 5. Local Element Vectors for Current and Previous Configurations**

555
ganathan 1989) in that element-based rather than nodal triad vectors are used to quantify the local displacements. This is necessary for the inclusion of large local rotations, which usually exist in plastic-hinge analysis (Izzuddin 1991).

At the last equilibrium configuration, the two vector sets, \( \{1c, 1c^o\} \) and \( \{2c, 2c^o\} \), representing the principal axes at the element ends, are normal to the chord vector \( c^o \), but not necessarily identical; the latter consideration depends on the cumulative twist of the element.

The vectors of principal axes are modified according to the increment of global rotations at the two ends, while the chord vector is only affected by translational displacements. Hence

\[
x^c = \begin{pmatrix} X^c_c \, Y^c_c \, Z^c_c \end{pmatrix}^T
\]

(22a)

\[
1c = \sum_{j=1}^3 1T_{i,j} 1c^o_j
\]

(22b)

\[
2c = \sum_{j=1}^3 2T_{i,j} \vec{c}^o_j
\]

(22c)

\[
1c = \sum_{j=1}^3 1T_{i,j} 1c^o_j
\]

(22d)

where

\[
X^c_c = X^c_c + u_2 - u_1
\]

(23a)

\[
Y^c_c = Y^c_c + v_2 - v_1
\]

(23b)

\[
Z^c_c = Z^c_c + w_2 - w_1
\]

(23c)

\[
L^c_c = \sqrt{(X^c_c)^2 + (Y^c_c)^2 + (Z^c_c)^2}
\]

(23d)

and

\[
1T = \begin{bmatrix} \alpha_1, \beta_1, \gamma_1 \end{bmatrix}
\]

(24a)

\[
2T = \begin{bmatrix} \alpha_2, \beta_2, \gamma_2 \end{bmatrix}
\]

(24b)

with \( T(\alpha, \beta, \gamma) \) having been defined in (8).

The inclination of the current principal axes relative to the chord axis determines the increment of relative element rotations. However, for the calculation of the increment of twist, a fictitious vector \( 2c^o \) is defined as the transformation of the principal \( y \)-axis at end 1 due to rotation increments at end 2. Assuming small incremental values, the increment of chord displacements \( \delta_c^o \) can be established according to the following relations:

\[
\delta_c^o = (\delta\theta_{1y}, \delta\theta_{1z}, \delta\theta_{2y}, \delta\theta_{2z}, \delta\Delta, \delta\theta_T)^T
\]

(25a)

\[
\delta\theta_{1y} = -x^c_i 1c^o_i c^e_i
\]

(25b)

\[
\delta\theta_{1z} = -x^c_i 1c^o_i c^e_i
\]

(25c)
\[ \delta \theta_{2y} = -x e^c \cdot \frac{2}{y} e^c \] .......................... (25d)
\[ \delta \theta_{2z} = -x e^c \cdot \frac{2}{z} e^c \] .......................... (25e)
\[ \delta \Delta = L^e - L^o \] .......................... (25f)
\[ \delta \theta_T = \frac{1}{2} e^c \cdot \frac{2}{y} e^c \] .......................... (25g)

where

\[ L^o = \sqrt{(X^o_E)^2 + (Y^o_E)^2 + (Z^o_E)^2} \] .......................... (26)

and

\[ \frac{2}{y} e_l = \sum_{j=1}^{3} T_{i,j} \frac{1}{y} e_j \] .......................... (27)

The current chord displacements \( \dot{c} u \) are obtained by updating the displacements of the last configuration

\[ \dot{c} u = c^o u + \delta \dot{c} u \] .......................... (28)

Once equilibrium is achieved, the principal axes are reset to a position normal to the current chord axis, so that the foregoing equations can be applied for the subsequent incremental step. This is performed according to the following procedure:

\[ x e^o = x e^c \] .......................... (29a)
\[ y e^o = \frac{1}{2} e^c \times \frac{1}{x} e^c \] .......................... (29b)
\[ z e^o = x e^c \times \frac{1}{y} e^c (y e^o \text{ after resetting}) \] .......................... (29c)
\[ \frac{2}{y} e^o = \frac{2}{z} e^c \times \frac{1}{x} e^c \] .......................... (29d)
\[ \frac{2}{z} e^o = \frac{2}{x} e^c \times \frac{1}{y} e^c (z e^o \text{ after resetting}) \] .......................... (29e)

**ELEMENT STRAIN ENERGY**

The solution of the equations of equilibrium, represented by (4), requires the definition of element global forces \( g f \), which are the first derivatives of the element strain energy \( \varepsilon_U \) with respect to the global displacements \( U \), as given by (5). However, since \( \varepsilon U \) is more conveniently expressed in terms of local displacements \( \dot{c} u \), chain differentiation rules are used to obtain \( g f \). An equivalent form of (5) that considers only the element global freedoms \( g u \) is given by

\[ g f_j = \frac{\partial (\varepsilon U)}{\partial _g u_j} = \sum_{i=1}^{6} T_{j,i} \frac{\partial (\varepsilon U)}{\partial \dot{c} u_i} \] .......................... (30)
\[ g f_j = \sum_{i=1}^{6} T_{j,i} e_i \] .......................... (31)

where
Vector \( \mathbf{f} \) represents local element forces that depend on the element type, as discussed in the next section. On the other hand, matrix \( \mathbf{T} \) is employed for the transformation of local to global element forces, and its terms can be explicitly obtained from (20)–(29), as given in Appendix I.

**LOCAL ELEMENT FORCES**

The local element forces \( \mathbf{f} \) referred to in the preceding section are first derivatives of the strain energy \( eU \) with respect to local displacements \( \mathbf{u} \), and depend on the element type.

For ordinary beam elements employing cubic and linear shape functions for the transverse and axial displacements, respectively, \( \mathbf{f} \) is given by the following relationship:

\[
\mathbf{f}_i = \sum_{j=1}^{6} \mathbf{k}_{i,j} \mathbf{u}_j \quad \text{(34)}
\]

where

\[
\mathbf{k}_i = \frac{1}{L} \begin{bmatrix}
4EI_y & 0 & 2EI_y & 0 & 0 & 0 \\
0 & 4EI_z & 0 & 2EI_z & 0 & 0 \\
2EI_y & 0 & 4EI_y & 0 & 0 & 0 \\
0 & 2EI_z & 0 & 4EI_z & 0 & 0 \\
0 & 0 & 0 & 0 & EA & 0 \\
0 & 0 & 0 & 0 & 0 & GJ
\end{bmatrix} \quad \text{(35)}
\]

In the current work, a quartic formulation, discussed in detail in Izzuddin (1991), is used for demonstrating the applicability of the developed procedures to the large-displacement analysis of space frames. The quartic formulation has eight degrees of freedom in the local system, as shown in Fig. 6, and provides the capability for modeling initial imperfections. The inclusion in this formulation of the additional midside transverse freedoms, accompanied by accounting for the effect of bowing on the centroidal axial strain, result in significant improvement in accuracy (Izzuddin 1991) over earlier beam-column formulations (Meek and Loganathan 1989; Jennings 1968).

**SOLUTION PROCEDURE**

The solution of the nonlinear system, represented by (4), requires the use of incremental iterative strategies, such as the family of Newton-Raphson procedures. For this purpose, an element global tangent stiffness matrix \( \mathbf{k} \) must be derived, with assembly procedures used to form the global tangent stiffness matrix of the overall structure.

The element global tangent stiffness \( \mathbf{k} \) can be obtained from (31) using chain differentiation properties, as follows:
FIG. 6. Local Freedoms of Quartic Formulation

\[
\begin{align*}
\mathbf{g}_{kij} &= \frac{\partial \mathbf{f}_i}{\partial \mathbf{u}_j} = \sum_{k=1}^{6} \left( \mathbf{T}_{i,k} \frac{\partial \mathbf{f}_k}{\partial \mathbf{u}_j} + \frac{\partial \mathbf{T}_{i,k}}{\partial \mathbf{u}_j} \mathbf{c}_k \right) \\
\mathbf{g}_{kij} &= \sum_{k=1}^{6} \left[ \mathbf{T}_{i,k} \left( \sum_{m=1}^{6} \frac{\partial \mathbf{c}_k}{\partial \mathbf{u}_m} \frac{\partial \mathbf{u}_m}{\partial \mathbf{u}_j} \right) + \frac{\partial \mathbf{T}_{i,k}}{\partial \mathbf{u}_j} \mathbf{c}_k \right]
\end{align*}
\]
but

\[
\frac{\partial f_k}{\partial u_m} = k_{k,m} \quad \cdots \quad (38a)
\]

\[
\frac{\partial u_m}{\partial u_j} = T_{j,m} \quad \cdots \quad (38b)
\]

\[
\frac{\partial T_{i,k}}{\partial u_j} = \frac{\partial^2 u_k}{\partial u_i \partial u_j} = G_{i,j,k} \quad \cdots \quad (38c)
\]

hence

\[
G_{i,j} = \sum_{k=1}^{6} \left[ \left( \sum_{m=1}^{6} T_{i,k} k_{k,m} T_{j,m} \right) + G_{i,j,k} f_k \right] \quad \cdots \quad (39)
\]

The stiffness contribution \(G_{i,j,k} f_k\) is referred to as geometric stiffness, since it reflects the effect of a change in geometry on the global forces. The geometric matrix \(G\) is a \(12 \times 12 \times 6\) matrix of second derivatives of local with respect to global displacements, the explicit terms of which can be obtained by differentiating once the terms of matrix \(T\) given in Appendix I (Izzuddin 1991).

**FIG. 7. Geometry and Loading of Circular Bend**
The previously described procedures have been implemented in ADAPTIC, a general-purpose computer program for the nonlinear static and dynamic analysis of steel, reinforced concrete, and composite frames (Izzuddin and Elnashai 1989). Three comparative examples are performed using ADAPTIC, and are presented herein to demonstrate the accuracy of the developed procedures.

**Circular Bend**

The 45° circular bend in Fig. 7 has been extensively used for the verification of finite-element formulations (Bathe and Bolourchi 1979; Surana et al. 1989).FIG. 8. Response Prediction of Circular Bend

**FIG. 9. Effect of Number of Load Increments on Response Prediction**

**EXAMPLES**

The previously described procedures have been implemented in ADAPTIC, a general-purpose computer program for the nonlinear static and dynamic analysis of steel, reinforced concrete, and composite frames (Izzuddin and Elnashai 1989). Three comparative examples are performed using ADAPTIC, and are presented herein to demonstrate the accuracy of the developed procedures.

**Circular Bend**

The 45° circular bend in Fig. 7 has been extensively used for the verification of finite-element formulations (Bathe and Bolourchi 1979; Surana et al. 1989).
and Sorem 1989). The bend is analyzed here using one and two imperfect quartic elements, and applying the load in 15 equal increments.

In Fig. 7

\[ k = \text{load factor} = \frac{P}{\left(\frac{EI}{R^2}\right)} \]  \hspace{1cm} (40)

and \( E = 10 \times 10^6 \text{ lb/in.}^2; \ v = 0.3; \ EI_y = EI_z = 8.333 \times 10^5 \text{ lb in.}^2; \) and
FIG. 12. Geometric Configuration of Elastic Dome

$GJ = 5.417 \times 10^5 \text{lb in.}^2$. With two elements, the quartic formulation gives identical results to those in Surana and Sorem (1989), as shown in Fig. 8. One quartic element still provides a very good comparison even though the bend curvature corresponds to high imperfection levels, thus compromising the assumption of small deformation relative to the element chord.

The bend is reanalyzed using two imperfect quartic elements, but applying the load in five increments only. The comparison in Fig. 9 demonstrates that the adopted three-dimensional incremental approach is almost insensitive to the size of the load increment applied.

**Lateral Torsional Buckling**

In this work, lateral torsional instability is not given a high priority, because of the complexities associated with extra warping freedoms and the modeling of warping in the presence of section plasticity. This is the main reason for neglecting the flexure-torsion coupling in the derivation of the
quartic formulation in the Eulerian system. However, in the absence of warping, such coupling can still be accounted for if a sufficient number of elements is used per member, since the effect of nodal displacements and rotations on geometry is accurately accounted for.

To demonstrate this, a thin rectangular cantilever strip is subjected to a semitangential moment \( M \) at its tip, and assumed to have an initial out-of-straightness of \( (L/1,000) \) to initiate lateral buckling, as shown in Fig. 10.

In Fig. 10

\[
M_b = \frac{\pi}{L} \sqrt{EI_y GJ} 
\]

\[
k = \text{load factor} = \frac{M}{M_b} 
\]

and \( E = 210 \times 10^9 \text{ N/m}^2; \ EI_y = 1.4 \times 10^6 \text{ N·m}^2; \ EI_z = 3.5 \times 10^3 \text{ N·m}^2; \ GJ = 5.2 \times 10^3 \text{ N·m}^2; \text{ and } \nu = 0.3. \)

The load-deflection response given by four quartic elements in Fig. 11 provides a good prediction of the theoretical buckling moment \( M_b \) (Argyris et al. 1978a, b). The buckling characteristics are still exhibited when using one quartic element, but the predicted response is overstiff, as expected, since the flexure-torsion coupling is neglected in the Eulerian system.

**Elastic Dome**

The framed dome of Fig. 12 was analyzed by Remseth (1979), who employed a Lagrangian beam-column formulation neglecting large global rotations. The same structure was later considered by Shi and Atluri (1988), who applied transformations that become erroneous when large nodal rotations are involved.

The static response to a concentrated load at the crown point is shown in Fig. 13, in which excellent agreement is demonstrated between the prediction of ADAPTIC employing one elastic quartic element per member.
and the results based on the formulation of Kondoh et al. (1986). The disagreement with the results of Shi and Atluri—and, more significantly, with those of Remseth—can be attributed to the point just mentioned regarding their consideration of large global rotations.

**CONCLUSION**

A new procedure was developed to model the effects of large displacements on the behavior of space frames. The development is entirely based on the principle of stationary total potential energy, with appropriate choice of reference systems allowing the procedure formalization in terms of transformations between the Eulerian and global systems. It was demonstrated that different modeling approaches of large rotations would lead to the same solution, provided that there are no applied moments.

Although an incremental approach was used to model the effect of large rotations, an improved rotational transformation matrix was adopted in order to reduce spurious lengthening of vectors upon rotation. Moreover, the calculation of rotations in the Eulerian system was performed incrementally using element-based vectors. Consequently, large relative rotations, which usually exist in plastic-hinge analysis, are readily accommodated.

Examples performed using the nonlinear analysis program ADAPTIC demonstrated the accuracy of the proposed procedure. It was also shown that the incremental strategy is almost insensitive to the size of the load step.

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**APPENDIX I. DERIVATION OF MATRIX T**

The transformation matrix $T$ referred to in (31) is a matrix of first derivatives of local with respect to global displacements. The $12 \times 6$ terms of $T$ can be established explicitly using (20)-(28), as shown here

\[ c \mathbf{u} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)^T \]  
\[ g \mathbf{u} = (u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2)^T \]  
\[ ^1\mathbf{A} = \{1, 2, 3\} = \text{numbers of translational global freedoms at end 1} \]  
\[ ^2\mathbf{A} = \{4, 5, 6\} = \text{numbers of rotational global freedoms at end 1} \]  
\[ ^2\mathbf{A} = \{7, 8, 9\} = \text{numbers of translational global freedoms at end 2} \]  
\[ ^2\mathbf{A} = \{10, 11, 12\} = \text{numbers of rotational global freedoms at end 2} \]  
\[ ^3\mathbf{A} = \{1, 2, 3\} \cup \{4, 5, 6\} = \text{numbers of translational global freedoms} \]  
\[ ^3\mathbf{A} = \{7, 8, 9\} \cup \{10, 11, 12\} = \text{numbers of rotational global freedoms} \]
\( a \mathcal{A} = \mathcal{A} \cup, \mathcal{A} = \text{numbers of all global freedoms} \) ................. (45g)

\[ T_{i,1} = \frac{\partial \theta_{ix}}{\partial g u_i} \] ................. (46a)

\[ T_{i,1} = -\frac{\partial x c^c}{\partial g u_i} \frac{1}{2} c^c \] \( i \in \mathcal{A} \) ................. (46b)

\[ T_{i,1} = -c^c \left( \frac{\partial^2 T_{1 c}}{\partial g u_i} \right) \] \( i \in \mathcal{A} \) ................. (46c)

\[ T_{i,1} = 0 \] \( i \in \mathcal{A} \) ................. (46d)

\[ T_{i,2} = \frac{\partial \theta_{iy}}{\partial g u_i} \] ................. (47a)

\[ T_{i,2} = -\frac{\partial x c^c}{\partial g u_i} \frac{1}{2} c^c \] \( i \in \mathcal{A} \) ................. (47b)

\[ T_{i,2} = -c^c \left( \frac{\partial^2 T_{2 c}}{\partial g u_i} \right) \] \( i \in \mathcal{A} \) ................. (47c)

\[ T_{i,2} = 0 \] \( i \in \mathcal{A} \) ................. (47d)

\[ T_{i,3} = \frac{\partial \theta_{iz}}{\partial g u_i} \] ................. (48a)

\[ T_{i,3} = -\frac{\partial x c^c}{\partial g u_i} \frac{1}{2} c^c \] \( i \in \mathcal{A} \) ................. (48b)

\[ T_{i,3} = 0 \] \( i \in \mathcal{A} \) ................. (48c)

\[ T_{i,3} = -c^c \left( \frac{\partial^2 T_{3 c}}{\partial g u_i} \right) \] \( i \in \mathcal{A} \) ................. (48d)

\[ T_{i,4} = \frac{\partial \theta_{iz^1}}{\partial g u_i} \] ................. (49a)

\[ T_{i,4} = -\frac{\partial x c^c}{\partial g u_i} \frac{1}{2} c^c \] \( i \in \mathcal{A} \) ................. (49b)

\[ T_{i,4} = 0 \] \( i \in \mathcal{A} \) ................. (49c)

\[ T_{i,4} = -c^c \left( \frac{\partial^2 T_{4 c}}{\partial g u_i} \right) \] \( i \in \mathcal{A} \) ................. (49d)

\[ T_{i,5} = \frac{\partial \Delta}{\partial g u_i} \] ................. (50a)

\[ T_{i,5} = \frac{\partial L c}{\partial g u_i} \] \( i \in \mathcal{A} \) ................. (50b)

\[ T_{i,5} = 0 \] \( i \in \mathcal{A} \) ................. (50c)
\[
T_{l,6} = \frac{\partial \theta_x}{\partial \mathbf{u}_i} \hspace{1cm} (51a)
\]
\[
T_{l,6} = 0 \quad i \in \mathcal{A} \hspace{1cm} (51b)
\]
\[
T_{l,6} = \left( \frac{\partial^2 \mathbf{T}}{\partial \mathbf{u}_i \partial \mathbf{e}^0} \right) \cdot \frac{21}{y} \mathbf{c} \quad i \in \mathcal{A} \hspace{1cm} (51c)
\]
\[
T_{l,6} = \frac{1}{2} \mathbf{e}^c \cdot \left( \frac{\partial^2 \mathbf{T}}{\partial \mathbf{u}_i \partial \mathbf{e}^0} \right) \quad i \in \mathcal{A} \hspace{1cm} (51d)
\]

where
\[
\frac{\partial \mathbf{e}_k^c}{\partial \mathbf{u}_i} = \left( \mathbf{x} \mathbf{c}_k^c \cdot \mathbf{x} \mathbf{c}_k^c - \mathbf{l}_{k,k} \right) \hspace{1cm} (52a)
\]
\[
\frac{\partial \mathbf{e}_k^c}{\partial \mathbf{u}_i} = \frac{\partial \mathbf{e}_k^c}{\partial \mathbf{u}_{i-6}} \hspace{1cm} (52b)
\]
in which \( \mathbf{I} = 3 \times 3 \) identity matrix; and
\[
\frac{\partial \mathbf{T}}{\partial \alpha} = \begin{bmatrix}
0 & \beta/2 & \gamma/2 \\
\beta/2 & -\alpha & -1 \\
\gamma/2 & 1 & -\alpha
\end{bmatrix} \hspace{1cm} (53a)
\]
\[
\frac{\partial \mathbf{T}}{\partial \beta} = \begin{bmatrix}
-\beta & \alpha/2 & 1 \\
\alpha/2 & 0 & \gamma/2 \\
-1 & \gamma/2 & -\beta
\end{bmatrix} \hspace{1cm} (53b)
\]
\[
\frac{\partial \mathbf{T}}{\partial \gamma} = \begin{bmatrix}
-\gamma & -1 & \alpha/2 \\
1 & -\gamma & \beta/2 \\
\alpha/2 & \beta/2 & 0
\end{bmatrix} \hspace{1cm} (53c)
\]
and
\[
\frac{\partial \mathbf{L}_c}{\partial \mathbf{u}_i} = -\mathbf{c}^c \hspace{1cm} i \in \mathcal{A} \hspace{1cm} (54a)
\]
\[
\frac{\partial \mathbf{L}_c}{\partial \mathbf{u}_i} = \frac{\partial \mathbf{L}_c}{\partial \mathbf{u}_{i-6}} \hspace{1cm} i \in \mathcal{A} \hspace{1cm} (54b)
\]

**APPENDIX II. REFERENCES**


APPENDIX III. Notation

The following symbols are used in this paper:

\( A = \) cross-sectional area;
\( a = \) vector in 3D space;
\( a' = \) vector in 3D space after rotation;
\( \mathbf{c} = \) vector of direction cosines of local x-axis of 3D element;
\( \mathbf{c}_1 = \) vector of direction cosines of local y-axis at end 1;
\( \mathbf{c}_2 = \) vector of direction cosines of local z-axis at end 1;
\( \mathbf{c}_3 = \) vector of direction cosines of local y-axis at end 2;
\( \mathbf{c}_4 = \) vector of direction cosines of local z-axis at end 2;
\( \mathbf{c}_5 = \) fictitious vector of direction cosines of local y-axis at end 1 due to rotations at end 2;
\( c = \) right-side superscript, denotes current values during an incremental step;
\( E = \) Young's modulus of elasticity;
\( \mathbf{f} = \) element basic local forces \( \langle M_{1y}, M_{1z}, M_{2y}, M_{2z}, F, M_T \rangle \);
\( \mathbf{f} = \) element global forces;
\( G = \) shear modulus;
\( G = \) element geometric stiffness matrix;
\( I_y = \) second moment of inertia in local y-direction;
\( I_z = \) second moment of inertia in local z-direction;
\( I = \) identity matrix;
\( J = \) St. Venant's torsion constant;
\( k = \) element local tangent stiffness matrix;
\( k = \) element global tangent stiffness matrix;
\( L = \) element length before deformation;
\( o = \) right-side superscript, denotes initial values during incremental step;
\( P = \) structure loading vector;
\( r = \) rotational vector in 3D space \( \langle \alpha, \beta, \gamma \rangle \);
\( T = \) right-side superscript, transpose sign;
\( T = \) matrix for transformation of global to local displacements;
\( r_T = \) rotational transformation matrix in 3D space;
\( T^1 = \) transformation due to rotations at end 1 of element;
\( T^2 = \) transformation due to rotations at end 2 of element;
\( U = \) strain energy;
\( U = \) global displacements of structure;
\( e_u = \) element basic local displacements \( \{\theta_{1y}, \theta_{1z}, \theta_{2y}, \theta_{2z}, \Delta, \theta_T\} \);
\( g_u = \) incremental element global displacements \( \{u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2\} \);
\( W = \) loss of potential energy of load;
\( X_E = \) element global X-axis coordinate;
\( Y_E = \) element global Y-axis coordinate;
\( Z_E = \) element global Z-axis coordinate;
\( \delta = \) incremental operator for variables, vectors, and matrices;
\( \Pi = \) total potential energy;
\( \Sigma = \) summation over range variable \( (i) \);
\( \langle \rangle = \) encloses terms of row vector;
\( |a| = \) magnitude of vector \( a \);
\( \cdot = \) vector dot product;
\( \times = \) vector cross product; and
\( \forall = \) “for all.”