03 Composite resistances

- **today.** YouTube: What is conduction, molecularly? simplified conduction equation story CENTER. Entrepreneurship resistances in series (like Ohm’s law)

- **announce.** Mid-term, Tues 03mar209, 6:30, 162 Willard Time, tonight 6:30, 307 Hammond

- **HW / quiz.** HW 02.Ch 02 Problems. quiz 03. next week.

- **pre-read.** still on P&S ch 02
What is conduction at the molecular level?

http://www.youtube.com/watch?v=77R4arwD8G8
Complete form of the conduction equation

\[
\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{q'''}{\rho c_v}
\]

\[
\alpha \equiv \frac{k}{\rho c_v}
\]

constant, uniform conductivity (k)

P&S p 16, eq 2.2
Simplification 1: No heat generation

\[
\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{q''}{\rho c_v}
\]

Fourier equation. No internal energy conversion, \(q'' = 0\).

\[
\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)
\]

This simplification is often called “the conduction equation”.
Simplification 2: Steady state

\[ \frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{q^\prime\prime}{\rho c_v} \]

**Poisson equation.** Steady state, with energy conversion. \( \partial / \partial t = 0. \)

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q^\prime\prime}{k} = 0 \]

Note that we have divided through by \( \alpha. \)
Simplification 3: No heat gen, and steady state

\[
\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{q''}{\rho c_p}
\]

Laplace equation. \( q'' = 0, \quad \partial / \partial t = 0. \)

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0
\]
Example: P&S 1.1 for T profile

Find the temperature profile for a 4.0 cm slab, with $T_1 = 38 \, \text{C}$, $T_2 = 21 \, \text{C}$, $k = 0.19 \, \text{W/m-K}$.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$
\[ \frac{d^2 T}{dx^2} = 0 \]

BC: \( x = x_1, \quad T = T_1 \)
\( x = x_2, \quad T = T_2 \)

\[ T = a + bx \]

\[ T_1 = a + bx_1 \]
\[ T_2 = a + bx_2 \]
\[ T_2 - T_1 = b(x_2 - x_1) \]
\[ b = \frac{T_2 - T_1}{x_2 - x_1} \]

\[ T = T_1 - (T_2 - T_1) \left( \frac{x - x_1}{x_2 - x_1} \right) \]
The conduction equation in Cartesian, Cylindrical

\[ \frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{q''}{\rho c_v} \]

**Cartesian:**

\[ \frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{q''}{\rho c_v} \]

**Cylindrical:**

\[ \frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{q''}{\rho c_v} \]
The conduction equation in spherical coordinates

\[
\frac{\partial T}{\partial t} = \alpha \left( \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial T}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \right) + \frac{q'''}{\rho c_v}
\]

In P&S they use $\psi$ for $\theta$. 
Story CENTER: Character. What do you see?

- Character
- Excellence
- Ownership
- Tenacity
- Entrepreneurship
- Relationship

http://www.janeelliott.com/
Example: P&S 1.1, but in cylindrical coordinates

Find the T profile and heat for \( r_1 = 0.5" \), \( r_2 = 0.5625" \), with \( T_1 = 38 \, \text{C} \), \( T_2 = 21 \, \text{C} \), \( k = 57 \, \text{W/m-K} \). The tube has a length \( L = 5 \, \text{ft} \). We are at steady state, no generation.

\[
\left\{ \begin{align*}
\frac{\partial T}{\partial t} &= \alpha \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) \\
0 &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right)
\end{align*} \right.
\]
T profile for cylindrical coordinates

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = 0
\]

BC 1: \( T(r_1) = T_1 \)
\( T(r_2) = T_2 \)

\[
\frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = 0
\]

\[
\frac{\partial T}{\partial r} = \frac{c_1}{r}
\]

\[
\Delta T = \frac{c_1 dr}{r}
\]

\[
T = c_1 \ln r + c_2
\]

\[
\frac{T - T_1}{T_2 - T_1} = \frac{ln \frac{r}{r_1}}{ln \frac{r_2}{r_1}}
\]
Steady-state heat in cylindrical coordinates

\[
\left\{ \begin{array}{l}
\frac{T-T_1}{T_2-T_1} = \frac{\ln \frac{r}{r_1}}{\ln \frac{r_2}{r_1}} \\
q = -kA \frac{\partial T}{\partial r}
\end{array} \right.
\]

\[
q = -\frac{2\pi k L (T_2 - T_1)}{\ln \frac{r_2}{r_1}}
\]

T = T_1 + (T_2 - T_1) \left[ \ln \frac{r}{r_1} - \ln \frac{r_2}{r_1} \right]

\[
\frac{dT}{dr} = \frac{(T_2 - T_1)}{\ln \frac{r_2}{r_1}} \left[ \frac{1}{r} \right]
\]

A = 2\pi r L
Steady-state heat in spherical coordinates

\[ q = \frac{4\pi k (T_2 - T_1)}{\frac{1}{r_2} - \frac{1}{r_1}} \]

P&S eq 2.14
Find the T profile and heat for \( r_1 = 0.5" \), \( r_2 = 0.5625" \), with \( T_1 = 38 \text{ C} \), \( T_2 = 21 \text{ C} \), \( k = 57 \text{ W/m-K} \). The tube has a length \( L = 5 \text{ ft} \). We are at steady state, no generation.

\[
\begin{align*}
q &= -\frac{2\pi k L (T_2 - T_1)}{ln \frac{r_2}{r_1}} \\
q &= +78.8 \text{ kW}
\end{align*}
\]
\[ Q = -kA \frac{dT}{dr} \]

\[ T = T_1 + (T_2 - T_1) \left( \frac{\ln r - \ln r_1}{\ln r_2/r_1} \right) \]

\[ \frac{dT}{dr} = \frac{T_2 - T_1}{\ln r_2/r_1} \cdot \frac{1}{r} \]

\[ A = 2\pi r L \]

\[ A \frac{dT}{dr} = \frac{2\pi L (T_2 - T_1)}{\ln r_2/r_1} \]

\[ q = -57 \text{ W} \times \frac{2\pi \times 5 \text{ ft}}{n \cdot \mu} \frac{m}{3.28 \text{ ft}} = \frac{(21 - 38 \text{ c})}{\ln \frac{0.5628}{0.5}} \]

\[ = +78.8 \text{ W} \]
Heat flow through one resistance (like Ohm’s law)

\[ I = \frac{V}{R} \]  
Ohm’s law

Current = \frac{potential}{resistance}

\begin{align*}
q &= -kA \frac{T_2 - T_1}{x_2 - x_1} \Delta x \\
q &= \frac{T_1 - T_2}{\Delta x/kA} = \text{driving potential/resistance}
\end{align*}
Heat flow through composite resistances

\[ I = \frac{V}{R} \]

Ohm’s law

\[ R = R_1 + R_2 \]

\[ q = \frac{T_1 - T_3}{\frac{\Delta x_a}{k_a A} + \frac{\Delta x_b}{k_b A}} \]

\[ \frac{\Delta x_a}{k_a A} \quad \frac{\Delta x_b}{k_b A} \]
How do we know we can just add resistances?

\[
q = -k_a A \frac{dT}{dx} \bigg|_a = -k_b A \frac{dT}{dx} \bigg|_b
\]

\[
q = -k_a A \frac{T_2 - T_1}{x_2 - x_1} = -k_b A \frac{T_3 - T_2}{x_3 - x_2}
\]

\[
T_2 - T_1 = -q \left( \frac{x_2 - x_1}{k_a A} \right)
\]

\[
T_3 - T_2 = -q \left( \frac{x_3 - x_2}{k_b A} \right)
\]
How do we know we can add resistances?

\[ T_2 - T_1 = -q \left( \frac{x_2 - x_1}{k_a A} \right) \]

\[ T_3 - T_2 = -q \left( \frac{x_3 - x_2}{k_b A} \right) \]

In the end, \( q \) is proportional to the \( T \) driving potential, and inversely proportional to the sum of the resistances.

\[ q = \frac{T_1 - T_3}{x_2 - x_1 + \frac{x_3 - x_2}{k_a A} + \frac{x_3 - x_2}{k_b A}} \]
\[ T_2 - T_1 = -q \left( \frac{X_2 - X_1}{h_c A} \right) \]

\[ T_3 - T_2 = -q \left( \frac{X_3 - X_2}{h_b A} \right) \]

\[ T_3 - T_1 = -q \left( \frac{X_2 - X_1}{h_c A} + \frac{X_3 - X_2}{h_b A} \right) \]

\[ q = \frac{T_1 - T_3}{ \frac{X_2 - X_1}{h_c A} + \frac{X_3 - X_2}{h_b A} + ( ) + ( )} \]
Cylindrical resistances in series

We can add as many in series as we wish.

\[ q = -\frac{2\pi L (T_3 - T_1)}{\frac{1}{k_a} \ln \frac{r_2}{r_1} + \frac{1}{k_b} \ln \frac{r_3}{r_2}} \]
Summary

\[ q = \frac{T_1 - T_3}{\Delta x_a / k_a A + \Delta x_b / k_b A} \]

What do you see?

Character
Excellence
Ownership
Tenacity
Entrepreneurship
Relationship
A = area for heat transfer [=] m²

\( b \) = body force [=] N/m³

\( c_v \) = constant volume heat capacity [=] J/kg-K; \( c = c_v \approx c_p \) for solid

\( g \) = gravitational acceleration [=] 9.8 m/s²

\( h \) = enthalpy per mass [=] J/kg; later used as “heat transfer coef”

\( k \) = thermal conductivity [=] W/m-K or BTU / h-ft-F

\( m \) = mass [=] kg or lbₘ (note to use gₙ conversion with lbₘ)

\( m \) = mass flowrate [=] kg/s when \( m \) has an overdot

\( p \) = pressure [=] Pa
List of symbols for Lectures 02-03

- \( q \) = heat transfer rate \([\text{=} W \text{ or BTU/h}]\)
- \( q''' \) = heat generation rate per volume \([\text{=} W/m^3]\)
- \( s \) = entropy per mass \([\text{=} J/kg-K]\)
- \( T \) = temperature \([\text{=} C \text{ or } K \text{ or } F \text{ or } R]\)
- \( u \) = internal energy per mass \([\text{=} J/kg]\)
- \( U \) = internal energy \([\text{=} J]\); usually used in HT for “overall HT coef”
- \( v \) = velocity, \( \nu \) = speed \([\text{=} m/s]\)
- \( V \) = volume \([\text{=} m^3]\)
- \( w \) = rate of work \([\text{=} W]\)
- \( x, y, z, r \) = distances in various coordinates \([\text{=} m]\)

\[
Q = \int_{0}^{t_f} q(t) \, dt
\]
List of symbols for Lectures 02-03

\( \alpha = k/\rho c = \text{thermal diffusivity} \ [\text{=} \ m^2/s] \)

\( \Delta = \text{change in, as in } \Delta x. \)

\( \theta (\psi \text{ in P&S}), \phi = \text{angles in various coordinates} \ [\text{=}] \ \text{none} \)

\( \eta = \text{viscosity} \ [\text{=}] \ \text{kg/m-s or Pa-s; } 1 \text{ cP} = 0.001 \ \text{Pa-s}. \)

\( \rho = \text{mass density} \ [\text{=}] \ \text{kg/m}^3 \)

\( \nabla = i_x \frac{\partial}{\partial x} + i_y \frac{\partial}{\partial y} + i_z \frac{\partial}{\partial z} \)

\( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \)
Opinion Box

- Take out a sheet of paper.
- Keep your paper anonymous, without your name.

Let me know …
- What made sense?
- What was confusing?
- Suggestions?
- Comments?

SPECIAL QUESTION
- Did you find the bio HW interesting? Tests?
- Did you find the portfolio exercise useful or helpful?