Weakly nonlinear cubic interactions between surface waves and interfacial waves: An analytic solution

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A set of third-order equations describing the interactions between surface waves and interfacial waves is presented. The specific interaction studied is that of exact subharmonic resonance of two interfacial waves by a single surface wave. The coupled amplitude evolution equations are solved analytically in terms of Jacobian elliptic functions. Upon specification of the initial conditions, the minimum and maximum amplitudes attained by the surface and interfacial waves, along with the nonlinear interaction period, are obtained. The results from the third-order theory are contrasted with those from second-order theory and highlight the importance of the cubic interactions.
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Nonlinear interactions between surface and interfacial waves were first studied by Ball.[1] In contrast to the work of Phillips[2] on deep-water surface waves, where a quartet of waves was required for resonance, the multiple branches of the dispersion relationship for a two-layer system were shown by Ball[1] to require only a triad of waves for resonance. The specific triad studied consisted of two surface waves and a single interfacial wave. Experimental investigations of this triad were conducted by Lewis *et al.*[3] and Joyce.[4]

A triad consisting of a single surface wave and two oblique subharmonic interfacial waves was more recently studied theoretically and experimentally by Hill and Foda[5] and Jamali.[6] Theoretically, these studies were limited to the second-order determination of the initial exponential growth rates of the interfacial waves. During this initial stage, the surface wave amplitude is assumed to be much larger than the interfacial wave amplitudes and, therefore, constant. This triad is fundamentally different from that studied by Ball[1] in that the interfacial waves now have frequencies that, while still less than, are now comparable to the frequency of the surface wave. Given the typically weak stratification encountered in most field situations, these frequencies correspond to quite short wavelengths.

The problem is therefore of interest because it suggests that large-amplitude, short-wavelength interfacial waves can be generated and maintained by a monochromatic surface wave field. Depending upon the parameters of the problem, such as surface wave amplitude, density ratio, etc., this interaction may be of sufficient strength to bring the interfacial waves to the point of breaking. If so, this could contribute to vertical mixing in stratified water bodies and to the transport of ''fluid mud'' in estuarine environments. Such breaking and mixing was observed in the experiments of Jamali,[6] which utilized a layer of fresh water over a layer of salt water. In the experiments of Hill and Foda,[5] which utilized a layer of light mineral oil over a layer of fresh water, breaking in the classic sense[7] was not observed, but significant mixing, as evidenced by droplets of water in the oil layer and vice versa, was observed.

The present Brief Communication extends the theoretical analysis of this triad by considering the later stages of development of the interfacial waves. As they grow, quadratic and cubic interactions begin to modify the surface wave field. Multiple-scales and perturbation analyses are used to obtain a coupled set of third-order amplitude evolution equations. In the limit of no damping, these equations can be solved exactly in terms of Jacobian elliptic functions. The benefit of this is that the maximum amplitudes attained by the interfacial waves and the nonlinear ''beating'' period of the interaction are explicitly determined.

Begin by assuming that a fluid layer of density ρ and depth H overlies a layer of density ρ' and great depth. Both layers are assumed to be inviscid. A Cartesian coordinate system is fixed at the interface with the y coordinate pointing upwards and the x and z coordinates defining the horizontal.

A surface wave of complex amplitude a_2, wavenumber k, and frequency ω is present and is propagating in the positive x direction. The interfacial wave field is made up of two waves that propagate at equal and opposite angles to the surface wave. In order to satisfy the conditions of exact subharmonic resonance, the x components of the wavenumber vectors of the two interfacial waves are both $k/2$ and the frequencies of both interfacial waves are $\omega/2$. The amplitudes of the interfacial waves are both a_1. The z components of the wavenumber vectors of the interfacial waves are $\pm l$ and the magnitudes of the interfacial wavenumber vectors are both therefore $\lambda=(l^2+k^2/4)^{1/2}$.

Next, while it is possible to solve the interaction problem for arbitrary layer depths, the results quickly become algebraically tedious. Thus, in addition to the lower layer being deep with respect to all three waves, the upper layer is taken to be deep with respect to the interfacial waves. Also, the

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perturbation framework of this analysis makes the implicit assumption that the nonlinearity is weak.

At the leading order, the displacements of the free surface and the interface, respectively, are given by

$$\eta = (a_2/2)\, e^{i(kx-\omega t)} + \text{c.c.},$$

$$\xi = (a_1/2)\, e^{i(kx/2+lz-\omega/2)} + (a_1/2)\, e^{i(kx/2-lz-\omega/2)} + \text{c.c.},$$

where c.c. denotes the complex conjugate. The linear harmonics are well-known and the limiting forms consistent with the above depth restrictions are easily derived from Lamb,[8] Art. 231. The dispersion relationship for the surface wave is $\omega^2 = gk$, where g is gravity, and, combining this with the dispersion relationship for the interfacial wave, it can be shown that

$$\lambda/k = (1+\gamma)/[4(1-\gamma)],$$

where $\gamma = \rho/\rho'$ is the density ratio.

At the second and third orders, quadratic and cubic interactions in the boundary conditions yield terms that are in phase with the fundamental harmonics. In order to remove these secular terms, the wave amplitudes must be allowed to vary slowly with time. For any given harmonic, the standard technique[9] of applying Green's theorem to the homogeneous and inhomogeneous solutions will then lead to the evolution equation for the amplitude of that harmonic. For the present problem, the evolution equations for the amplitudes of the interfacial and surface wave fields are found to be

$$\dot{a}_1 = i\alpha_1 a_1^* a_2 + i\beta_{11} a_1 |a_1|^2 + i\beta_{12} a_1 |a_2|^2, \tag{1}$$

$$\dot{a}_2 = i\alpha_2 a_1^2 + i\beta_{21} a_2 |a_1|^2 + i\beta_{22} a_2 |a_2|^2, \tag{2}$$

where the dot notation indicates derivatives with respect to time and the α and β terms are given by

$$\alpha_1 = -\frac{\omega^3(3\gamma^2 - 2\gamma + 3)e^{-kH}}{8g(1+\gamma)^2},$$

$$\beta_{11} = -\frac{\omega k^2(6 + 6\gamma^2 - 13\gamma)}{32(1-\gamma)^2},$$

$$\beta_{12} = -\tfrac{13}{32}\omega k^2 e^{-2kH},$$

$$\alpha_2 = \frac{\omega k(3\gamma^3 - 5\gamma^2 + 5\gamma - 3)}{4(1+\gamma)^2[2\gamma\sinh(kH) + e^{-kH}]},$$

$$\beta_{21} = -\frac{\omega k^2(25\gamma^2 - 46\gamma + 25)}{32(1-\gamma)(\gamma e^{2kH} + 1 - \gamma)},$$

$$\beta_{22} = -\frac{\omega k^2(e^{-3kH}(1-\gamma) + \gamma e^{kH})}{2[2\gamma\sinh(kH) + e^{-kH}]}.$$

To facilitate solution of these equations, the complex amplitudes are represented as

$$a_1 = A_1 e^{i\theta_1}, \quad a_2 = A_2 e^{i\theta_2},$$

where the A and θ terms are now real quantities. Upon this substitution, and the definition $\Theta = 2\theta_1 - \theta_2$, the evolution equations are recast as

$$\dot{A}_1 = \alpha_1 A_1 A_2 \sin\Theta, \tag{3}$$

$$\dot{A}_2 = -\alpha_2 A_1^2 \sin\Theta, \tag{4}$$

$$\dot{\Theta} = [2\alpha_1 A_2 - \alpha_2(A_1^2/A_2)]\cos\Theta + (2\beta_{11} - \beta_{21})A_1^2 + (2\beta_{12} - \beta_{22})A_2^2. \tag{5}$$

While it is straightforward to numerically integrate these equations, it is also possible to solve them exactly.[10–12] With the following definitions and scalings

$$w = -\frac{A_1^2}{2\alpha_1} - \frac{A_2^2}{2\alpha_2}, \quad u = \frac{A_1}{\sqrt{-2\alpha_1 w}},$$

$$v = \frac{A_2}{\sqrt{-2\alpha_2 w}}, \quad \tau = -\alpha_1\sqrt{-2\alpha_2 w}\, t,$$

the evolution equations are recast as

$$\dot{u} = -uv\sin\Theta, \tag{6}$$

$$\dot{v} = u^2\sin\Theta, \tag{7}$$

$$\Theta = \dot{\overline{\ln(u^2 v)}}\cot\Theta + pu^2 + qv^2, \tag{8}$$

where the dot notation now indicates derivatives with respect to the slow time scale τ and p and q are constants given by

$$p = (2\beta_{11} - \beta_{21})\sqrt{-(2w/\alpha_2)},$$

$$q = (2\beta_{12} - \beta_{22})\sqrt{-(2\alpha_2 w/\alpha_1^2)}.$$

In addition to the well-known Manley–Rowe relation, $u^2 + v^2 = 1$, it is possible to derive an additional constant of the motion

$$u^2 v\cos\Theta - (p/4)u^4 + (q/4)v^4 = \Gamma. \tag{9}$$

Finally, these constants are incorporated into the evolution equations to yield the single equation

$$(\dot{\overline{v^2}})^2 = 4\left\{(1-v^2)^2 v^2 - \left[\Gamma + \frac{p}{4}(1-v^2)^2 - \frac{q}{4}v^4\right]^2\right\} = f(v^2). \tag{10}$$

At this point a distinction between two possible cases must be made. If the four roots of $f(v^2) = 0$ are real, denote them as $v_d^2 \geq v_c^2 \geq v_b^2 \geq v_a^2$. The solution to (10) is then given by

$$v^2 = v_d^2 - \frac{(v_d^2 - v_a^2)(v_d^2 - v_b^2)}{v_d^2 - v_b^2 + (v_b^2 - v_a^2)\text{sn}^2[n(\tau + \tau_0), m]}, \tag{11}$$

where m and n are given by

$$n^2 = (v_c^2 - v_a^2)(v_d^2 - v_b^2)\,[(p-q)^2/16],$$

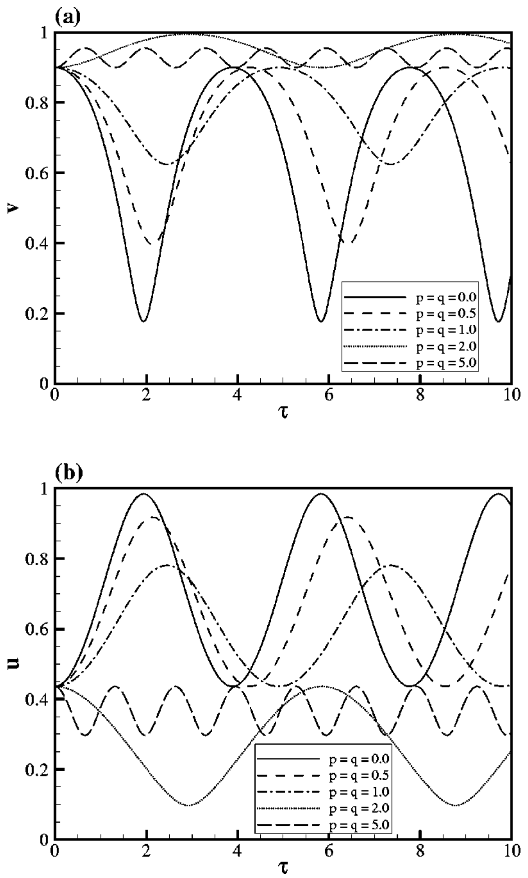

FIG. 1. Transient evolution of (a) v and (b) u for the conditions $v_i = 0.9$, $\Theta_i = 0.0$, and various values of the nonlinearity parameters p and q.

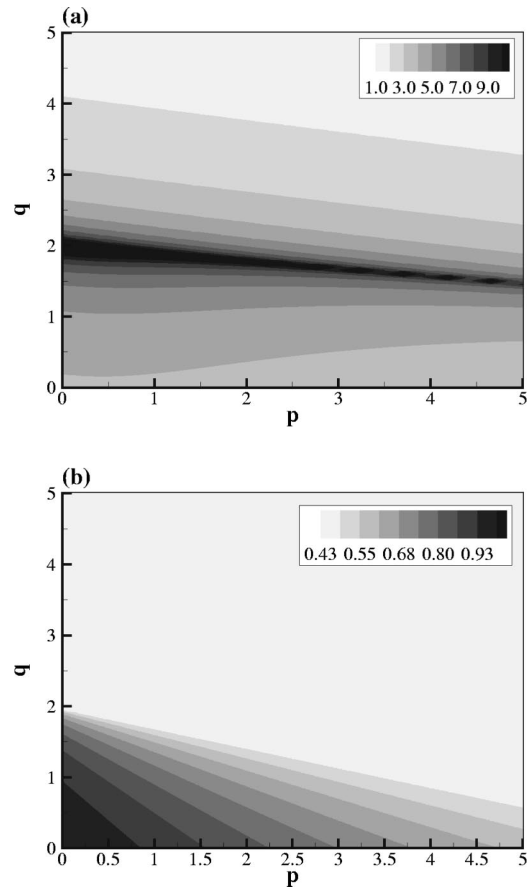

FIG. 2. Contours of (a) interaction period T and (b) $u_{\max}$ in (p,q) parameter space for the conditions $v_i = 0.9$ and $\Theta_i = 0.0$.

$$m^2 = [(v_d^2 - v_c^2)(v_b^2 - v_a^2)]/[(v_d^2 - v_b^2)(v_c^2 - v_a^2)],$$

and τ_0 is determined from the initial condition on v. It is clear, therefore, that v^2 oscillates between v_a^2 and v_b^2 with a period given by

$$T = 2K(m)/n, \tag{12}$$

where K is the complete elliptic integral of the first kind. Upon conversion back to dimensional values, the maximum and minimum amplitudes of the surface and interfacial waves, along with the beat period of the nonlinear interaction are obtained.

The other case corresponds to two real roots and a complex conjugate pair. In this case, denote the two real roots as $v_b^2 \geqslant v_a^2$ and represent the complex conjugate pair as $a \pm bi$. Following Weiland and Wilhelmsson,[11] introduce

$$G_1 = \sqrt{(v_b^2)^2 - 2av_b^2 + a^2 + b^2},$$

$$G_2 = \sqrt{(v_a^2)^2 - 2av_a^2 + a^2 + b^2}.$$

In this case, the solution is given by

$$v^2 = \frac{\dfrac{2(v_b^2 - v_a^2)G_1G_2}{(G_1 - G_2)^2}}{\dfrac{2G_2}{G_1 - G_2} + 1 - \mathrm{cn}[n(\tau + \tau_0), m]} + \frac{v_a^2 G_1 - v_b^2 G_2}{G_1 - G_2}, \tag{13}$$

where the parameters n and m are now defined by

$$n^2 = G_1G_2[(p-q)^2/16],$$

$$m^2 = \frac{1}{2}\left\{1 - \frac{1}{G_1G_2}[v_a^2 v_b^2 - a(v_a^2 + v_b^2) + a^2 + b^2]\right\}.$$

Again, v^2 oscillates between v_a^2 and v_b^2 with a period given by (12).

Figure 1 illustrates the temporal evolution of v and u for several values of the nonlinearity parameters p and q and for the initial conditions $\Theta_i = 0$, $v_i = 0.9$. Noting that $p = q = 0$ corresponds to the second-order theory, it is evident that increasing third-order nonlinearity serves to restrict the growth of the interfacial waves. Also, the interaction period first lengthens, and then decreases with increasing nonlinearity. These trends are more fully illustrated in (p,q) parameter space in Fig. 2. Contours of the nonlinear interaction period are given in Fig. 2(a) and contours of the maximum value attained by the interfacial mode (i.e., $u_{\max}$) are given in Fig. 2(b).

The obvious line in Fig. 2(a) where the interaction period tends to infinity is a special case where all of the energy is asymptotically transferred from the fundamental harmonic (interfacial wave) to the second harmonic (surface wave). In this case, $m \to 1$, and the elliptic functions sn and cn are

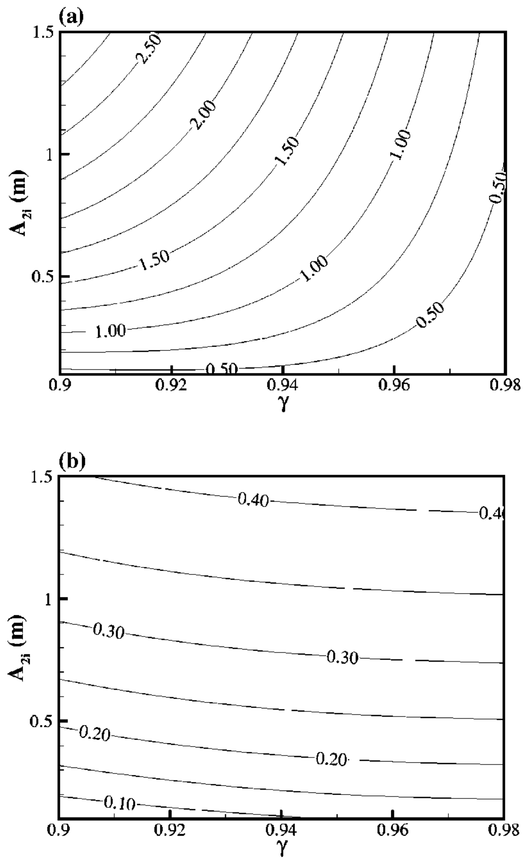

FIG. 3. Contours of (a) maximum amplitude $2A_{1_{\max}}$ and (b) maximum steepness $2lA_{1_{\max}}$ of resonated interfacial wave field as functions of density ratio γ and initial surface wave amplitude A_{2_i}. $H=20$ m, $T=12$ s, $A_{1_i}=0.01$ m.

replaced by the hyperbolic tangent and secant functions, respectively. It is readily shown that this occurs along the line in (p,q) space defined by

$$q=[(4v_i\cos\Theta_i)/(91+v_i^2)]-[(1-v_i^2)/(1+v_i^2)]\,p.$$

The net effect of selecting other values for the initial conditions v_i and Θ_i will be to alter the slope and intercept of this line.

From a practical point of view, a major question of interest is whether or not this resonance mechanism is strong enough to break interfacial waves at typical field scales. It should be stressed that the weakly nonlinear methodology of the present analysis cannot, by itself, describe wave breaking. Indeed, the only question that can answered here is whether or not an interfacial wave field can be forced to a steepness where breaking is expected to occur. For guidance in setting a threshold steepness, recall that Thorpe,[7] in his studies of standing interfacial waves, found that breaking at the nodes was first observed at a steepness of around 0.2 and was clearly obvious at a steepness of 0.4. While it is not immediately obvious that this result applies to the present interfacial wave field, which is standing in one direction and progressive in another, a representative critical steepness of 0.3 will nevertheless be adopted. Moreover, since the lateral component of the interfacial wavenumber vector is, given moderate to weak stratification, much greater than the streamwise component (i.e., $l\gg k/2$), a breaking criterion of $2lA_{1_{\max}}>0.3$ will be used. The factor of two takes into account the fact that the interfacial wave field is made up two interfacial wave trains, each having amplitude A_1.

Consider surface swell with a period of 12 s propagating on an upper layer that is 20 m deep. Figure 3 shows, as a function of the layer density ratio γ and initial surface wave amplitude A_{2_i}, the third-order predictions for the maximum amplitude of the interfacial wave field ($2A_{1_{\max}}$) and the corresponding maximum steepness ($2lA_{1_{\max}}$). The maximum interfacial amplitude is observed to be a strongly decreasing function of the density ratio, but the dependence of the wave steepness on the density ratio is observed to be fairly weak. In order to exceed the critical steepness of 0.3, a surface wave amplitude of approximately 80 cm is required.

In stark contrast to these results, calculations based only upon quadratic interactions show the interfacial amplitude to be a strongly increasing function of density ratio. Moreover, the second-order calculations significantly over-predict the interfacial wave amplitude; in the same (γ,A_{2_i}) parameter space as shown in Fig. 3, the second-order theory predicts maximum amplitudes of up to 15 m, whereas the third-order theory predicted maximum amplitudes only up to 3 m.

In conclusion, this analysis has shown that surface waves have the ability, through a subharmonic resonance mechanism, to generate large-amplitude interfacial waves. The analytic solution of the evolution equations fully determines the maximum and minimum values of the waves and the nonlinear interaction period. The results indicate that the algebraic temptation to truncate the analysis at the second order will significantly overestimate the maximum interfacial amplitude.

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