1.1. Outline of Lecture

1.2. Concept of Shear Dispersion

To this point, we have considered how mass is transported by the processes of molecular diffusion and turbulent diffusion. In the latter case, turbulent eddies are responsible for the observed spread of tracer. In this lecture, we wish to consider the effect that lateral velocity gradients have on the mixing process. We will find that these gradients cause significantly greater mixing than turbulent diffusion alone. This process of ‘shear dispersion’ is conceptually illustrated in Fig. 1.

In the upper portion of a figure, an initial patch of tracer has been distributed across the cross section of a channel. In this channel, the lateral profile of longitudinal velocity is uniform. As a result, as the dye is advected downstream, there is a noticeable widening of the cloud. This is due solely to the effects of longitudinal turbulent diffusion.

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Figure 1. Diffusion and dispersion in channels with and without lateral velocity gradients.

In the lower portion of the figure, an initial patch of tracer has been distributed across a channel in which there is lateral variation in the streamwise velocity. This type of velocity profile is what we expect to occur in the presence of boundary shear. As the dye cloud is advected downstream, it is distorted due to the velocity profile. The consequence of this distortion is that strong lateral concentration gradients are created. These large gradients, in turn, drive significant lateral mixing. As a result, the cloud of tracer that is subject to this ‘shear dispersion,’ will experience much greater longitudinal mixing than the cloud that is experiencing longitudinal diffusion only.

It is important to recognize that shear dispersion is not really a ‘new’ mixing process. Rather, it is nothing more than lateral diffusion operating in conjunction with lateral velocity shear.

1.3. Derivation of the Advection-Dispersion Equation

Field experiments on longitudinal mixing processes quickly reveal many of the characteristics that we are familiar with, namely concentration profiles that decrease in amplitude and increase in width. Given that we associate these behaviors with gradient models of the mixing process, we can immediately conjecture that shear dispersion will be well-modeled by a similar gradient hypothesis. However, it is not immediately obvious why this should be so.
To proceed, we must introduce the idea of laterally averaging variables such as velocity and concentration. We are already familiar with averaging through our study of turbulence. In that case, however, the averaging was with respect to time. Now, we express variables as

$$u(y) = \bar{u} + u'(y),$$

where the overbar denotes the average and we are considering averaging in the vertical direction. The same type of average can be defined for the transverse ($z$) direction. The mean value is given by

$$\bar{u} = \frac{1}{h} \int_0^h u \ dy,$$

where $h$ is the flow depth. We call $u'$ a deviation rather than a fluctuation as we did in the case of Reynolds averaging. This highlights the fact that primed values will be known. For example, if we are looking at uniform, turbulent, open-channel flow, the velocity profile is known. Hence, the mean is easily derived and the deviations from the mean are immediately known at all depths.

Next, let us establish the conditions for our example. We will consider two-dimensional (no transverse velocity gradients; only vertical) flow and we will consider the dispersion of a ‘plane’ of dye that is introduced across the cross-section of our channel. Thus, the advection-diffusion equation is

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial y^2}.$$

Note that the variables in this equation are implicitly assumed to be time-averaged quantities. We do not use the overline notation to denote this, as we will be reserving it to denote the vertical average described above.

Substituting into this equation our variable decompositions, we have

$$\frac{\partial (\bar{C} + C')}{\partial t} + (\bar{u} + u') \frac{\partial (\bar{C} + C')}{\partial x} = D_x \frac{\partial^2 (\bar{C} + C')}{\partial x^2} + D_y \frac{\partial^2 (\bar{C} + C')}{\partial y^2}.$$

Next, we operate on the premise that longitudinal dispersion will greatly exceed longitudinal diffusion. Additionally, we note that $u' \neq f(x)$ and that $\bar{C} \neq f(z)$. Thus, our equation simplifies to

$$\frac{\partial (\bar{C} + C')}{\partial t} + (\bar{u} + u') \frac{\partial (\bar{C} + C')}{\partial x} = D_y \frac{\partial^2 C'}{\partial y^2}.$$
At this point, we, as we have done before, introduce a coordinate transformation that puts into a frame of reference moving at the depth-averaged velocity \( \bar{u} \). This is accomplished by introducing \( \xi = x - \bar{u}t \). We recall that this change requires that care be taken in carrying our the derivatives in the equation. Most notably is the change

\[
\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - \bar{u} \frac{\partial}{\partial \xi}.
\]

As a result, the modified equation is

\[
\frac{\partial (\bar{C} + C')}{\partial t} + u' \frac{\partial (\bar{C} + C')}{\partial \xi} = D_y \frac{\partial^2 C'}{\partial y^2}.
\]

1.3.1. Taylor’s Analysis

This equation is still far too complicated for us to deal with. As a result, Taylor considered justifiable simplifications that could be made. First of all, consider the term

\[
u' \frac{\partial (\bar{C} + C')}{\partial \xi}.
\]

If we are considering dispersion in the relatively far-field, we expect that the vertical variation of tracer is fairly weak. This amounts to

\[
u' \frac{\partial C'}{\partial \xi} \ll u' \frac{\partial \bar{C}}{\partial \xi}.
\]

Second, Taylor hypothesized that, at fairly long times, the concentration deviations would attain a relatively steady state. The net effect of these two lines of reasoning is that our equation becomes

\[
u' \frac{\partial \bar{C}}{\partial \xi} = D_y \frac{\partial^2 C'}{\partial y^2}.
\]

1.4. Solution of the Advection-Dispersion Equation

This equation is easily integrated twice, yielding

\[
C'(y) = \frac{1}{D_y} \frac{\partial \bar{C}}{\partial \xi} \int_0^y \int_0^y u' \, dy' \, dy.
\]

While we are nearly finished, we need to briefly reconsider the issue of mass flux in our channel. The mass flux, which is a function of depth is given by

\[
q = u' (\bar{C} + c').
\]
If we depth-average this, we obtain

\[ \bar{q} = \frac{1}{h} \int_{0}^{h} u'(C + C') \, dy. \]  

Recalling that the average of a single fluctuation is, by definition, zero, this simplifies to \( \bar{q} = \bar{u}'\bar{C}' \).

If we substitute into this result our result for \( C' \), we have

\[ \bar{q} = \frac{1}{h} \int_{0}^{h} u' \frac{1}{D_y} \frac{\partial C}{\partial \xi} \int_{y}^{y} \int_{y}^{y} u' \, dy \, dy \, dy. \]  

Since \( \frac{\partial \bar{C}}{\partial \xi} \neq f(y) \), we rewrite this as

\[ \bar{q} = -K \frac{\partial \bar{C}}{\partial \xi}, \]  

where

\[ K = -\frac{1}{h} \int_{0}^{h} u' \frac{1}{D_y} \int_{0}^{y} \int_{0}^{y} u' \, dy \, dy \, dy. \]

Remarkably, therefore, we have shown that longitudinal dispersion results in mass flux in the form of a gradient hypothesis! Even more remarkably, we have an equation that will allow us to explicitly calculate dispersion coefficients based upon knowledge of the turbulent diffusion coefficient and the velocity profile.

Before moving on to consider some example, we recall our knowledge of conservation of mass. The accumulation of mass in our control volume is balanced by the longitudinal gradient in tracer mass flux. In our moving frame of reference, this is given by

\[ \frac{\partial \bar{C}}{\partial t} = -\frac{\partial \bar{q}}{\partial \xi} = K \frac{\partial^2 \bar{C}}{\partial \xi^2}. \]

Reverting to our original frame of reference yields

\[ \frac{\partial \bar{C}}{\partial t} + \bar{u} \frac{\partial \bar{C}}{\partial x} = K \frac{\partial^2 \bar{C}}{\partial x^2}. \]

Thus, shear dispersion behaves exactly like turbulent advection diffusion and we can immediately make use of the solutions that we obtained during our consideration of that process.

To briefly summarize, we have accomplished the following:

- We have shown that mass flux in advection-dispersion is proportional to the gradient in the mean concentration.
- We have derived an explicit equation for the dispersion coefficient.
We have shown that advection-dispersion is governed by the same equation as advection-diffusion.

As a caveat, however, recall that Taylor made numerous assumptions in leading us to these results. One of these is that a certain amount of time must elapse before dispersion behaves as a gradient model. Specifically, he argues that \( t > \frac{0.4h^2}{D} \) is required. For low, molecular diffusivities, this may be an inordinate amount of time. For real rivers, where turbulence is present, however, this time requirement is more moderate. To revisit the Waikato River example of Rutherford, recall that we found a diffusivity of 0.01 \( \text{m}^2 \text{s}^{-1} \). Based upon this, and the average depth of 2.6 \( \text{m} \), a startup time on the order of a few minutes is required.

1.5. Dispersion Coefficients

While we will focus on engineering estimates for \( K \) in a subsequent lecture, it is of interest to consider sample calculations for \( K \) based upon simple velocity profiles. More complex velocity profiles, based for example on real field data, can be handled through numerical integration.

**Exercise 1.** Consider first the laminar two-dimensional flow of a fluid between two parallel plates separated by a gap of \( h \). If the lower plate is fixed and the upper plate is moved to the right at speed \( U \), a simple shear flow with a linear velocity profile is established. If the origin of the \( y \) coordinate is positioned at the bottom of the gap, the velocity profile is given by \( u = \frac{Uy}{h} \). The deviation is therefore given by \( u' = \frac{Uy}{h} - \frac{U}{2} \).

Calculation of the dispersion coefficient is straightforward, if lengthy:

\[
K = -\frac{1}{h} \int_0^h \int_0^y \int_0^y u' \frac{1}{D_y} dy dy dy
\]

\[
= -\frac{1}{h} \int_0^h \int_0^y \int_0^y \left( \frac{Uy}{h} - \frac{U}{2} \right) dy dy dy
\]

\[
= -\frac{1}{h} \int_0^h \int_0^y \frac{Uy^2}{2h} - \frac{Uy}{2} dy dy
\]

\[
= -\frac{1}{h} \int_0^h \left( \frac{Uy}{h} - \frac{U}{2} \right) \frac{1}{D_y} \left( \frac{Uy^3}{6h} - \frac{Uy^2}{4} \right) dy
\]

\[
= \frac{U^2h^2}{120D_y}.
\]
As a numerical example, consider a thin layer of fluid that is set into motion by wind shear stress. If $U = 0.01 \text{ m s}^{-1}$, $h = 0.1 \text{ m}$ and $D = 10^{-9} \text{ m}^2 \text{ s}^{-1}$, we obtain a dispersion coefficient of around $10 \text{ m}^2 \text{ s}^{-1}$, billions of times greater than the molecular diffusivity!

1.5.1. Application to Turbulent OCF

This exact same process may be applied to turbulent shear flows. Recall that we have already discussed the vertical variation of velocity, shear, and diffusivity in a turbulent open channel flow. While the process is straightforward in concept, it is more complicated mathematically. Elder (1959) is credited with being the first to execute the analysis and his work showed that

\begin{equation}
K = 5.93hu_*.
\end{equation}

Recall that we found the vertical turbulent diffusion coefficient to be $0.067hu_*$ and that Rutherford suggests that longitudinal and vertical turbulent diffusion are similar. Thus, the conclusion is that longitudinal dispersion, due to velocity shear, is two orders of magnitude more effective at mixing than longitudinal diffusion due to turbulence.