Midterm I. Math 230. Spring 2005

4, October, 2006

**Problem 1** Pick a point on the first line. Say, \( t = 0 \) yields \( A(0, 1, 0) \in L_1 \). Pick a point on the second line. Say, \( s = 0 \) yields \( B(1, 4, 0) \in L_2 \). Take any direction vector of either line, say, \( \vec{d} = (2, -3, 1) \) is the direction vector of the second line. Vectors \( \vec{d} \) and \( \vec{AB} \) are parallel to the plane of the two lines, but not to each other. Therefore, \( \vec{n} = \vec{d} \times \vec{AB} \) is orthogonal to the plane, and so can be taken as its normal vector. \( \vec{n} = \vec{d} \times \vec{AB} = \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ 1 & 3 & 0 \end{vmatrix} = (-3, 1, 9). \)

To define a plane we also need some point on it, say, \( A(0, 1, 0) \), so the equation is \(-3(x - 0) + 1(y - 1) + 9(z - 0) = 0\), or just \(-3x + y + 9z - 1 = 0\).

**Problem 2** \( z = \rho \cos(\phi) = 2\cos(t) = 2\sqrt{3} \)
\( r = \rho \sin(\phi) = 2\sin(t) = 2 \frac{\sqrt{3}}{2} = 1 \)
\( x = r \cos(\theta) = 1 \cos(\frac{\pi}{3}) = \frac{1}{2} \)
\( y = r \sin(\theta) = 1 \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2} \)

So the answer to (a) is \( (\sqrt{3}, \frac{\sqrt{3}}{2}, \frac{1}{2}) \).
\( \rho^2 \sin^2(\phi) = (\rho \sin(\phi))^2 = r^2 = x^2 + y^2 \)
\( \rho^2 \cos^2(\phi) = (\rho \cos(\phi))^2 = z^2 \)

So the answer to (b) is \( 2(x^2 + y^2) - z^2 = 1. \)

**Problem 3** \( \vec{T'}(t) = (e^t, -e^{-t}, \sqrt{2}) \)
\( |\vec{T'}(t)| = \sqrt{e^{2t} + e^{-2t} + 2} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t} \)

Hence, the unit tangent vector \( \vec{T}(t) = \frac{1}{e^t + e^{-t}}(e^t, -e^{-t}, \sqrt{2}) \)

Its derivative can be found applying the product rule (derivative of a product of a scalar function and a vector function):
\[ \vec{T}'(t) = \frac{1}{(e^t + e^{-t})^2} (e^t, -e^{-t}, \sqrt{2}) + \frac{1}{e^t + e^{-t}} (e^t, -e^{-t}, \sqrt{2})^{-1} (e^t, -e^{-t}, 0) = \left( \frac{-e^t e^{-t}}{(e^t + e^{-t})^2} \right) (e^t, -e^{-t}, \sqrt{2}) + \frac{1}{(e^t + e^{-t})^2} (e^{2t} + 1, 1 + e^{-2t}, 0) = (-e^{2t} + 1, 1 - e^{-2t}, -\sqrt{2e^t + \sqrt{2}e^{-t}}) + \frac{1}{(e^t + e^{-t})^2} (e^{2t} + 1, 1 + e^{-2t}, 0) = \frac{1}{(e^t + e^{-t})^2} (2, 2, -\sqrt{2e^t + \sqrt{2}e^{-t}}) \]

We need to normalize \( \vec{T}'(t) \) in order to find the unit normal vector. \( |\vec{T}'(t)| = \frac{1}{(e^t + e^{-t})^2} \sqrt{4 + 4 + 2(e^{2t} - e^{-t})^2} \), so the unit normal vector \( \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{1}{\sqrt{4 + 4 + 2(e^{2t} - e^{-t})^2}} (2, 2, -\sqrt{2e^t + \sqrt{2}e^{-t}}) \)
Problem 4 $\mathbf{\tau}'(t) = \langle t, \sqrt{2t}, 1 \rangle$

$|\mathbf{\tau}'(t)| = \sqrt{2t^2 + 2t + 1} = t + 1$, as far as $1 \leq t \leq 3$, so $t + 1 \geq 0$. Finally,

$L = \int_1^3 |\mathbf{\tau}'(t)| dt = \int_1^3 (t + 1) dt = \left(\frac{t^2}{2} + t\right) \bigg|_1^3 = \left(\frac{3^2}{2} + 3\right) - \left(\frac{1^2}{2} + 1\right) = 6$

Problem 5 $\mathbf{\tau}(t) = \mathbf{\tau}(0) + \int_0^t \mathbf{\tau}'(u) du = \langle 0, 1, 2 \rangle + \langle (-\sin(u), \cos(u), u^2 - u) \rangle \bigg|_0^t = \langle 0, 1, 2 \rangle + \langle (-\sin(t), \cos(t), t^2 - t) - \langle 0, 1, 0 \rangle \rangle = \langle -\sin(t), \cos(t), t^2 - t + 2 \rangle$

$\mathbf{\tau}(t) = \mathbf{\tau}(0) + \int_0^t \mathbf{\tau}'(u) du = \langle 0, 1, 2 \rangle + \langle (\cos(u), \sin(u), u^2 - u^2 + 2u) \rangle \bigg|_0^t = \langle 1, 0, 0 \rangle + \langle (\cos(t), \sin(t), t^2 - t^2 + 2t) - \langle 1, 0, 0 \rangle \rangle = \langle \cos(t), \sin(t), \frac{t^2}{2} - \frac{t^2}{2} + 2t \rangle$

$|\mathbf{\tau}(1)| = |\langle -\sin(1), \cos(1), 1^2 - 1 + 2 \rangle| = \sqrt{(-\sin(1))^2 + (\cos(1))^2 + (2)^2} = \sqrt{5}$

Problem 6 We did not cover this material

Problem 7 We did not cover this material

Problem 8 Extensive computations. Idea: substitute one equation into the other. $x + y + x^2 + 3y^2 = 4$, or $(x + 1/2)^2 + 3(y + 1/6)^2 = 13/3$. This projection onto the $xy$-coordinate plane is an ellipse, which can be parametrized by $t$. Then find the tangent vector (i.e. the derivative of the vector function, describing this parametrization). Finally, substitute the value of $t$ yielding $(1, -1, 4)$. This value of $t$ can be found from the parametrization of the ellipse mentioned above, i.e. from the first coordinate. $t = \cos^{-1}(...)$.

Problem 9 True: $a, e$ (check the dot product), $h$ (think of the volume of the corresponding parallelepiped), $i$

False: $b$ (because they may lie on one line), $c, d$ (in our definition curvature is always nonnegative), $f$ (the center is $(2, -3, 5)$), $j$ ($f_{xy}$ means the opposite, but you do not need to know that for the exam)

Statement $g$ makes no sense, because this is a cross product of a vector with a scalar.

Problem 10 The question is in effect to find the tangential component of the acceleration. $\mathbf{\tau}'(t) = \mathbf{\tau}''(t) = \langle \frac{1}{2}, 2t, 2t \rangle$

$\mathbf{\tau}'(t) = \mathbf{\tau}'(t) = \langle \frac{1}{2}, 2, 2 \rangle$

$\text{proj}_{\mathbf{\tau}'(t)} \mathbf{\omega}(t) = \frac{\mathbf{\omega}(t) \cdot \mathbf{\tau}'(t)}{|\mathbf{\tau}'(t)|^2} \mathbf{\tau}'(t) = \frac{\frac{1}{2} + 4t + 4t^2}{\frac{1}{4} + 4t^2 + 4t^2} \langle \frac{1}{2}, 2t, 2t \rangle = \frac{4t^2 - 1}{8t^2 + 1} \langle \frac{1}{2}, 2, 2 \rangle$

Problem 11 We did not cover this material