Answers to Quiz 3

Math 230. Friday, 9/22/6

Problem 1 (75%) Convert \((1, \frac{\pi}{2}, -1)\) from cylindrical to rectangular and spherical coordinates.

Of course, one way is to draw a picture and to determine the coordinates from the picture. However, it is better to find the coordinates analytically and then to use the picture for verification.

Given cylindrical coordinates \((1, \frac{\pi}{2}, -1)\), i.e. \(r = 1, \theta = \frac{\pi}{2}, z = -1\). Converting to rectangular coordinates. \(x = r \cos(\theta), y = r \sin(\theta)\). As far as \(\cos(\frac{\pi}{2}) = 0, \sin(\frac{\pi}{2}) = 1\), \(x = 1 \cdot 0 = 0, y = 1 \cdot 1 = 1\), thus the rectangular coordinates of this point are \((0, 1, -1)\). (\(z\) is the same as in the cylindrical coordinates).

Secondly, find the spherical coordinates. A typical mistake was to confuse \(\rho\) with \(r\). \(\rho\) is the distance from the origin (the pole) to the point, while \(r\) is the distance from the origin to the projection of the point onto the \(xy\)-coordinate plane. \(\rho\) can be found from the rectangular coordinates by the distance formula \(\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0^2 + 1^2 + (-1)^2} = \sqrt{2}\) or from the cylindrical coordinates \(\rho = \sqrt{r^2 + z^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}\). \(\theta\) is the same as in the cylindrical coordinates. How to find \(\phi\)? It is known that \(0 \leq \phi \leq \pi\), therefore \(\phi\) is uniquely defined by its cosine or tangent, but not by its sine. Recalling that \(z = \cos(\phi)\rho\), the cosine can be found: \(\cos(\phi) = \frac{z}{\rho} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}\). Therefore, as far as \(0 \leq \phi \leq \pi\), \(\phi = \cos^{-1}(-\frac{\sqrt{2}}{2}) = \frac{3\pi}{4}\).

Answer: \((\sqrt{2}, \frac{\pi}{2}, \frac{3\pi}{4})\). Note that, even though \(\sin(\phi) = \frac{\sqrt{2}}{2}, \tan(\phi) = -1\), but \(\sin^{-1}(\frac{\sqrt{2}}{2}) = \tan^{-1}(-1) = \frac{\pi}{4}\). Therefore the best is to check the result by converting it back from spherical to rectangular or cylindrical coordinates. A picture can also be helpful.

Problem 2 (25%) Classify the surface: \((1 + x)(1 - x) = (y + z)(y - z)\).

\((a + b)(a - b) = (a - b)(a + b) = a^2 - ba + ab - b^2 = a^2 - b^2\). In particular, \((1 + x)(1 - x) = 1 - x^2, (y + z)(y - z) = y^2 - z^2\). Therefore, the equation now looks like \(1 - x^2 = y^2 - z^2\), or \(x^2 + y^2 - z^2 = 1\). This is the canonical equation of the hyperboloid of one sheet. Without this knowledge, still certain observations can be made. For instance, in this equation only squares of the coordinates appear: \(x^2, y^2, z^2\), therefore the surface is symmetric with respect
to all three coordinate planes. Hence it is neither an elliptic paraboloid nor a hyperbolic paraboloid. The $x$- and $y$- axes are in the same relation to the surface, but the $z$-axis is different. Thus the surface is neither a sphere nor an ellipsoid. The trace $x = 0$ gives a hyperbola $y^2 - z^2 = 1$. The trace $z = 0$ is a circle $x^2 + y^2 = 1$, which a hyperbolic cylinder presumably cannot have.

Finally, we need to distinguish between the hyperboloid of one sheet and the hyperboloid of two sheets. This hyperboloid is oriented along the $z$-axis. Clearly, it is centered at the origin (being symmetric with respect to all three coordinate planes). If it were of two sheets, it would not intersect the plane $z = 0$. However, in this case even in the cross-section by the plane $z = 0$ we see a unit circle, therefore this hyperboloid is of one sheet.