Answers to Quiz 4

Math 230. Friday, 9/29/6

Problem 1 (75%) Find arc length. \( x = \sin(e^t), y = \cos(e^t), z = 1, -1 \leq t \leq 2. \)

"Find arc length" means to find a particular positive real number, not the arc length function. This vector function is merely a different parametrization of an arc of the unit circle. However, this observation is not necessary for solving the problem.

\[
L = \int_{-1}^{2} |\vec{r}'(t)| \, dt
\]

\[
\vec{r}' = \frac{d}{dt} (\sin(e^t), \cos(e^t), 1) = (e^t \cos(e^t), e^t \cdot (-\sin(e^t)), 0)
\]

\[
|\vec{r}'| = \sqrt{(e^t \cos(e^t))^2 + (e^t \cdot (-\sin(e^t)))^2 + (0)^2} = \sqrt{e^{2t} \cdot (\cos^2(e^t) + \sin^2(e^t))} = \sqrt{e^{2t}} = e^t
\]

\[
L = \int_{-1}^{2} |\vec{r}'| \, dt = \int_{-1}^{2} e^t \, dt = e^t \bigg|_{-1}^{2} = e^2 - e^{-1}
\]

Note that \( e^{-1} = \frac{1}{e} \neq 0. \) Note that if we replace the vector function \( \vec{r} = (\sin(e^t), \cos(e^t), 1) \) with \( \vec{r} = (1000 + \sin(e^t), 1000 + \cos(e^t), 1000 + 1), \) the answer (arc length) will not change. Of course, the derivative \( \vec{r}' \) will not change either. However, \( \int_{-1}^{2} |\vec{r}'(t)| \, dt \) will change a lot. Therefore, this integral has nothing to do with the arc length of this curve.

Problem 2 (25%) Find the osculating plane of the curve \( x = y^2, z = 0 \) at the point \( (1, 1, 0) \).

This curve is a plane curve, a parabola in particular. The osculating plane of a plane curve at any point is the plane containing this curve — in this case, the \( xy \)-coordinate plane (the plane \( z = 0 \)).

To find formally, the parabola should be parametrized by \( t \). There are two natural parametrizations: \( \vec{r}_1(t) = (t, \sqrt{t}, 0) \) and \( \vec{r}_2(t) = (t^2, t, 0). \) The latter is simpler, so let \( \vec{r}(t) = \vec{r}_2(t) = (t^2, t, 0). \) The point \( (1, 1, 0) \) corresponds to the value \( t = 1. \)
Taking the derivative, $\mathbf{r}'(t) = (2t, 1, 0)$. In particular, $\mathbf{r}'(1) = (2, 1, 0)$. $|\mathbf{r}'(t)| = \sqrt{4t^2 + 1}$. In particular, $|\mathbf{r}'(1)| = \sqrt{5}$. Normalizing, obtain the unit normal vector. $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{4t^2 + 1}}(2t, 1, 0) = \left(\frac{2t}{\sqrt{4t^2 + 1}}, \frac{1}{\sqrt{4t^2 + 1}}, 0\right)$. In particular, $T(1) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$ — is a unit vector, as expected.

The next step is to find $\mathbf{T}'(t)$. Note that $T'(t)$ IS NOT parallel to $\mathbf{r}''(t)$, and one cannot use the latter instead. It is crucial that $\mathbf{T}(t)$ is always a unit vector, and therefore $\mathbf{T}'(t)$ is always orthogonal to $\mathbf{T}(t)$. On the contrary, $\mathbf{r}''(t)$ (which can be viewed as the acceleration of a particle) may or may not be orthogonal to $\mathbf{r}'(t)$ (which may be viewed as the velocity of a particle). Of course, $\mathbf{r}'(t)$ is parallel to $\mathbf{T}(t)$, but $\mathbf{T}'(t)$ still has to be found.

$\mathbf{T}'(t) = \frac{d}{dt}\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{4t^2 + 1}}(2t, 1, 0) \neq \frac{1}{\sqrt{5}}(2, 0, 0)$. The whole thing has to be differentiated. Namely, $\mathbf{T}'(t) = \frac{d}{dt}(\frac{2t}{\sqrt{4t^2 + 1}}, \frac{1}{\sqrt{4t^2 + 1}}, 0) = \left(\frac{2(4t^2 + 1)^{\frac{3}{2}} - 2(2t)(4t^2 + 1)^{-\frac{1}{2}}}{4t^2 + 1}, \frac{2(4t^2 + 1)^{-\frac{1}{2}}}{4t^2 + 1}, 0\right)$. $N(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$, but there is no need to express it formally. It suffices to substitute $t = 1$: $\mathbf{T}'(1) = \left(\frac{2}{5\sqrt{5}}, \frac{1}{5\sqrt{5}}, 0\right)$. So $|\mathbf{T}'(1)| = \frac{2}{5}$, and $N(1) = \frac{5}{2}\left(\frac{2}{5\sqrt{5}}, \frac{1}{5\sqrt{5}}, 0\right) = \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0\right)$. This is a unit vector and it is orthogonal to $\mathbf{T}(1) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$, as expected (one can easily check the dot product).

One can finally find $\mathbf{B}(1) = \mathbf{T}(1) \times N(1) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \times \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0\right) = (0, 0, -1)$. Thus, the equation of the osculating plane at $(1, 1, 0)$ has the form $0(x - 1) + 0(y - 1) - 1(z - 0) = 0$, which is essentially $z = 0$.

How to avoid excessive computations? There was no need to find $N(1)$. It was enough to use $\mathbf{T}'(1)$, because it has the same direction. Thus, $\mathbf{T}(1) \times \mathbf{T}'(1)$ is also a normal vector to the osculating plane at $(1, 1, 0)$, and can as well be used to find the equation of the plane.

Moreover, before starting to differentiate $\mathbf{T}(t)$ to find $\mathbf{T}'(t)$ one could have observed that the $z$-coordinate of both vectors is zero. Therefore, both $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are parallel to the $xy$-coordinate plane for all $t$, and thus so is $N(t)$. Hence, the cross product of $\mathbf{T}$ and $\mathbf{N}$ must be perpendicular to the $xy$-plane and so be either $(0, 0, 1)$ or $(0, 0, -1)$ (because $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is also a unit vector). Both yield the same equation of a plane.

Finally, it is also possible to find $N(1)$ without much computations. This should be a horizontal vector with the $z$-coordinate equal to zero, as discussed above. It is known to be a unit vector, orthogonal to $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$. There are only two such vectors, $\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0\right)$ and $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$. It is easy to graph a parabola and to see, how is directed the normal vector $(1, 1)$. (When graphing, note that this is the parabola $x = y^2$, not $y = x^2$.)

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