Answers to Quiz 6

Math 230. Friday, 10/20/6

There were two problems, both graded out of 4 points. The best score was then chosen.

Problem 1 (4 points) *Use linear approximation to estimate* $\ln(1.02+\sin(0.05))$

Let $f(x,y)=\ln(x+\sin(y))$. Note that it is necessary to introduce f explicitly. You can't write f_x without specifying, what f is. Then we need to approximate $f(1.02,0.05)$. It is easy to notice that $1.02\approx 1, 0.05\approx 0$. However, this does not imply that $f(1.02,0.05)\approx f(1,0)$ — this approximation will not be linear.

f is differentiable, $f_x(x,y)=\frac{1}{x+\sin(y)}$, $f_y(x,y)=\frac{\cos(y)}{x+\sin(y)}$. The linear approximation of f near $(1,0)$ is given by

$$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \tag{1}$$

where $x_0=1, y_0=0$. $f(x_0,y_0)=f(1,0)=\ln(1+\sin(0))=\ln(1+0)=\ln(1)=0$. Avoid writing $f(x,y)=f(1,0)=0$ in this context, because in general $f(x,y)\neq 0$, so it misleads and confuses the reader. Evaluating, $f_x(x_0,y_0)=f_x(1,0)=\frac{1}{1+\sin(0)}=1$, $f_y(x_0,y_0)=f_y(1,0)=\frac{\cos(0)}{1+\sin(0)}=\frac{1}{1}=1$. It is important to use $f_x(x_0,y_0), f_y(x_0,y_0)$, and not $f_x(x,y), f_y(x,y)$. The main idea of the approximation is to choose one point where it is easy to evaluate f, f_x, f_y, and then use these values to approximate $f(x,y)$ for $(x,y)\approx(x_0,y_0)$.

Finally, using (1), $f(1.02,0.05)\approx L(1.02,0.05)=f(1,0)+f_x(1,0)(1.02-1)+f_y(1,0)(0.05-0)=0+1\cdot 0.02+1\cdot 0.05=0.07$.

There is a different solution. It is possible to approximate first $\sin(0.05)\approx\sin(0)+\sin'(0)(0.05-0)=0+1\cdot 0.05=0.05$, i.e. $\sin(0.05)\approx 0.05$, and then to proceed with $\ln(1.02+\sin(0.05))\approx\ln(1.02+0.05)=\ln(1.07)\approx 0.07$, because $\ln(1+0.07)\approx\ln(1)+\ln'(1)(1.07-1)=0+\frac{1}{1}\cdot 0.07=0.07$. This approximation is also valid and yields the same result, because the approximation is linear, i.e. **it works only for the first differentials**. Conclusion: if you don't understand whether you can approximate a particular composite function by composing the approximations of its arguments or not, do not do that.

The justification is as follows (you may skip it): $f(1.02,0.05)\approx f(1,0)+df$. Using the Chain rule (for differentials), $d(\ln(x+\sin(y))=\frac{1}{x+\sin(y)}d(x+\sin(y))$. $d(x+\sin(y))=dx+d(sin(y))=dx+\frac{dy}{\cos(y)}$. Hence, $\ln(1.02+\sin(0.05))\approx$

$\ln(1) + \frac{1}{1+\sin(0)} \cdot (0.02 + \frac{0.05}{\cos(0}) = 0 + (0.02 + 0.05) = 0.07$. This works only for the *first* differentials.

Problem 2 (4 points) $f(x) = \cos(x - \sin(x - e^x))$. $h(s,t) = f(s-t)$. $h_s(s,t) + h_t(s,t) =?$

There were two main types of mistakes while solving this problem. The first one was - meaningless partial derivatives. f is a function of one variable x. $\frac{df}{dx} = f'(x)$ makes sense, while $\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial h}, \frac{\partial f}{\partial y}$ make no sense. h is a function of two variables s and t. Therefore, $h_s = \frac{\partial h}{\partial s}$ and $h_t = \frac{\partial h}{\partial t}$ make sense, while $\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial f}, h'(x), h'(s), h'(t)$ make no sense.

The second problem was: parentheses. **Mind the parentheses!** More than 50% of grades for this problem decreased because of them. Make sure that you have the same number of opening and closing braces in each expression, and that the grouping is correct. If you omit braces occasionally, then expressions tend to be reinterpreted incorrectly. For instance, $(2 + 2 \cdot (2 + 2) =?$ It can be 16 or 10...

This problem is a typical application of the chain rule. A straightforward solution is possible. $h(s,t) = f(s-t) = \cos(s - t - \sin(s - t - e^(s - t)))$. Then h can be formally differentiated with respect to s and t, and these two partial derivatives added up. However, there is no need in it.

h can be viewed as a composite function. $x = x(s,t) = s - t$, $f(x) = \cos(x - sin(x - e^x))$. Then $h(s,t) = f(x(s,t))$. Therefore, $\frac{\partial h}{\partial s} = f'(x(s,t))\frac{\partial x(s,t)}{\partial s} = f'(s-t)\frac{\partial(s-t)}{\partial s} = f'(s-t) \cdot (1) = f'(s-t)$. Similarly, $\frac{\partial h}{\partial t} = f'(s-t)\frac{\partial(s-t)}{\partial t} = f'(s-t) \cdot (-1) = -f'(s-t)$. Hence, $h_s(s,t) + h_t(s,t) = f'(s-t) - f'(s-t) = 0$, now matter what f is (as far as it is differentiable).

Exercise: explain this result geometrically, in terms of level curves and the gradient of $h(s,t)$.