Math 251  Spring 2002  Midterm Exam 1

1. (b)

2. (d) because between equilibrium solutions \( y = 0 \) and \( y = 1 \) the right-hand side of the equation is negative, therefore the solution starting at \( \frac{9}{10} \) must decrease to a nearest equilibrium solution, which is 0.

3. (d) since we need to check when
\[
\frac{\partial}{\partial y} (e^x \sin y + bx^2 y^2) = \frac{\partial}{\partial x} (e^x \cos y + x^3 y)
\]

4. (c)

5. (d) because after dividing the whole equation by \( t^2 - 1 \) we will get discontinuities of the coefficients at \(-1\) and \(1\), and the initial point 5 lies to the right of the last point of discontinuity.

6. 
\[
\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = -1
\]

\[
ydy = -xdx
\]

\[
\frac{1}{2} y^2 = -\frac{1}{2} x^2 + \text{const}
\]

\[
y^2 = -x^2 + \text{const}
\]

\[
(-1)^2 = -0^2 + \text{const}
\]

\[
\text{const} = 1
\]

\[
y(x) = -\sqrt{-x^2 + 1}
\]

7. 
\[
t^2 \frac{dy}{dt} + 3ty = e^t, \quad t > 0
\]

\[
\frac{dy}{dt} + \frac{3}{t} y = \frac{1}{t^2} e^t
\]

\[
\mu(t) = e^{\int \frac{3}{t} dt} = e^{3 \ln t} = t^3
\]

\[
y(t) = \int \frac{t^3}{t^3} e^t dt = \frac{te^t - e^t + C}{t^3} = \frac{1}{t^2} e^t - \frac{1}{t^3} e^t + \frac{C}{t^3}
\]

8. First note that the equation is exact. Now
\[
\psi = \int (2xy - 3x^2) dx = x^2 y - x^3 + C(y)
\]

\[
\frac{\partial}{\partial y} (x^2 y - x^3 + C(y)) = x^2 + C'(y) = x^2 + 2y
\]

\[
C'(y) = 2y
\]

\[
C(y) = y^2 + \text{const}
\]

\[
\psi = x^2 y - x^3 + y^2 = \text{const}
\]
9. (a) 
\[ y'' - 4y' + 4y = 0 \]
\[ \lambda^2 - 4\lambda + 4 = 0 \]
\[ \lambda_{1,2} = 2 \]
\[ y(t) = C_1e^{2t} + C_2te^{2t} \]

(b) 
\[ y'' - 4y' + 5y = 0 \]
\[ \lambda^2 - 4\lambda + 5 = 0 \]
\[ \lambda_{1,2} = 2 \pm i \]
\[ y(t) = C_1e^{2t}\cos t + C_2e^{2t}\sin t \]

10. 
\[ y'' - 4y = 0 , \quad y(0) = 4 \quad y'(0) = 4 \]
\[ \lambda^2 - 4 = 0 \]
\[ \lambda_1 = 2 \quad \lambda_2 = -2 \]
\[ y(t) = C_1e^{2t} + C_2e^{-2t} \]
\[ y(0) = C_1 + C_2 = 4 \]
\[ y'(0) = 2C_1 - 2C_2 = 4 \]
\[ C_1 = 3 \quad C_2 = 1 \]
\[ y(t) = 3e^{2t} + e^{-2t} \]

11. 
\[ \frac{dQ}{dt} = 2 \cdot 5 - 2 \cdot \frac{Q(t)}{2}, \quad Q(0) = 15 \]
\[ \frac{dQ}{dt} + Q(t) = 10 \]
\[ Q(t) = Ce^{-t} + 10 \]
\[ Q(0) = C + 10 = 15 \]
\[ C = 5 \]
\[ Q(t) = 5e^{-t} + 10 \]
\[ Q(ln 5) = 5e^{-ln 5} + 10 = 11 \]
12.

\[ m \frac{dv}{dt} = mg - 2v \quad v(0) = 0 \]

\[ \frac{dv}{dt} + v = 10 \]

\[ v(t) = Ce^{-t} + 10 \]

\[ v(0) = C + 10 = 0 \]

\[ C = -10 \]

\[ v(t) = -10e^{-t} + 10 \]

\[
\text{distance} = \int_{0}^{2} v(t) dt = \int_{0}^{2} (-10e^{-t} + 10) dt = (10e^{-t} + 10t) \bigg|_{0}^{2} = 10 + 10e^{-2}
\]