A 3D Parabolic Equation Method for Sound Propagation in Moving Inhomogeneous Media

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In this paper, a new formulation of the Helmholtz equation for sound propagation in a moving inhomogeneous medium in cylindrical coordinates is derived. Based on this formulation, a novel three-dimensional parabolic equation is constructed. This parabolic equation can be used to model sound propagation in an inhomogeneous arbitrary moving medium. The method is used here to simulate three-dimensional outdoor sound propagation with an arbitrary wind above a rigid flat ground surface. The numerical results are presented and compared with benchmark analytical results to validate the methodology. Examples of propagation problems are then included, that demonstrate the importance of modeling the wind directly, rather than using an effective speed of sound.

Nomenclature

\[ c \] 
Local speed of sound

\[ c_o \] 
Reference speed of sound

\[ c_1, c_2, ..., c_9 \] 
Coefficients of Eqn. 2

\[ i \] 
\[ \sqrt{-1} \]

\[ k \] 
Reference wavenumber

\[ M \] 
Mach number

\[ p \] 
Complex acoustic pressure

\[ p_1, p_2, q_1, q_2 \] 
Coefficients of the Padé approximation for the square root operator, Eqn. 14

\[ R \] 
Distance between the acoustic source and the observer locations

\[ r, \theta, z \] 
Polar cylindrical coordinates

\[ v \] 
Wind velocity

\[ v_i \] 
Wind velocity components

\[ x \] 
\( (x, y, z) \)
Cartesian coordinates

\[ x_i \] 
Coordinate system components

\[ \omega \] 
Angular frequency of the sound source

\[ \rho \] 
Local medium density

\[ \rho_0 \] 
Reference/ambient medium density

\[ \psi \] 
\( \psi = \sqrt{(r)}p \exp(-ikr) \)

I. Introduction

The frequency domain Parabolic Equation (PE) method has been used widely for noise propagation predictions due to its computational efficiency. Lentovich and Fock\(^1\) originally introduced the PE method for electromagnetic wave propagation. Tappert\(^2\) introduced this technique to the ocean acoustics community in

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the 1970’s. Its main purpose was to predict long-range, low frequency sound propagation in range-dependent environments. Considerable developments have been made over the last three decades and these have been reviewed by Lee et al.\textsuperscript{3}

In most previous research, two-dimensional (2D) models have been used to represent three-dimensional (3D) acoustic propagation problems using an axisymmetric approximation. It has been assumed that the 2D models were sufficient if azimuthal variations in the medium were sufficiently weak. However, previous ocean acoustics research\textsuperscript{4,5} has found that 3D effects should not necessarily be neglected and 3D models are required in many situations. Thus, several 3D models have been constructed for underwater acoustics research.\textsuperscript{4–9} In these models the effect of a mean current is considered through the use of an effective sound speed approximation, in which the current velocity component in the propagation direction is added to the local speed of sound. Consequently, when the current direction is perpendicular to the sound propagation direction, the current velocity is treated as zero. Furthermore, even if the current velocity is parallel to the sound propagation direction, the effective sound speed approach is just an approximation, which is only effective if certain conditions are satisfied.\textsuperscript{10} Since the sound speed is usually much larger than the magnitude of the current velocity for underwater acoustics, this approximation can be effective. In cases where the medium velocity is a more significant fraction of the sound speed, such as in outdoor sound propagation, a PE model including the wind velocity explicitly should improve its accuracy.

Ostashev\textsuperscript{11,12} developed a wave equation in a moving medium with arbitrary inhomogeneities in sound speed and density. Based on this equation, a parabolic equation for three-dimensional wave propagation in Cartesian coordinates was obtained.\textsuperscript{13} This included the wind velocity explicitly. However, long-range sound propagation for a point source involves spherical wave propagation, which cannot be represented easily in Cartesian coordinates. Therefore, a parabolic equation written in cylindrical coordinates is more desirable. In the present paper, a 3D parabolic equation for sound propagation in cylindrical coordinates for a moving inhomogeneous medium is derived. The resulting equation is then solved using the Crank-Nicholson finite difference method. Numerical simulations are conducted for several three-dimensional atmospheric sound propagation problems with different wind profiles above a flat rigid ground surface. The numerical results are compared with benchmark analytical results to validate the methodology. Examples of propagation problems are then included that demonstrate the importance of including the wind directly.

\section{II. Governing equations}

\subsection{A. The wave equation for a moving inhomogeneous medium in cylindrical coordinates}

The reduced wave equation (Helmholtz equation) for a sound field \( p(x) \) propagating in an inhomogeneous arbitrary moving medium can be written in Cartesian coordinates as,\textsuperscript{11}

\[
\left[ \nabla^2 + k^2(1 + \epsilon) - \left[ \nabla \ln(\rho/\rho_0) \right] \cdot \nabla - \frac{2i}{\omega} \frac{\partial v_1}{\partial x} \right] \frac{\partial^2 p}{\partial x^2} + \frac{2ik}{c_0} \mathbf{v} \cdot \nabla \right] p(x) = 0
\]

In this equation, Cartesian tensor notation is used. \( x_1 \) is generally taken to be the direction of wave propagation and \( x_3 \) is the vertical or height direction. \( \epsilon = c_0^2/c^2 - 1 \) is the deviation from unity of the square of the refraction index in a motionless medium. Eqn. (1) is derived from a full set of linearized fluid dynamic equations with three assumptions: 1) internal gravity waves are neglected; 2) the mean flow is incompressible, that is, \( \nabla \cdot \mathbf{v} = 0 \); and 3) terms of order \( \mu^2 = \max(\nu^2/c_0^2, |\nu/\rho_0|, |\nu/\rho_0|) \) are ignored, where \( \tau = c - c_0 \) and \( \tau = \rho - \rho_0 \) are the deviations of the sound speed and density from their mean values.\textsuperscript{11}

Eqn. (1) can be transformed from Cartesian coordinates into cylindrical polar coordinates. In cylindrical coordinates \((r, \theta, z)\), \( r \) and \( \theta \) define a horizontal plane. \( r \) is the range, \( \theta \) is the azimuthal direction, and \( z \) is the height. The wave equation in cylindrical coordinates from Eqn. (1) can be written as,

\[
\left[ c_1 \frac{\partial^2}{\partial r^2} + k^2(1 + \epsilon) + c_2 \frac{\partial}{\partial r} + c_3 \frac{\partial}{\partial \theta} + c_4 \frac{\partial^2}{\partial \theta^2} + c_5 \frac{\partial^2}{\partial z^2} + c_6 \frac{\partial^2}{\partial r \partial z} + c_7 \frac{\partial^2}{\partial \theta \partial z} + c_8 \frac{\partial^2}{\partial r \partial \theta} + c_9 \frac{\partial}{\partial z} \right] p(r, \theta, z) = 0
\]

(2)

where, \( c_1, c_2, ..., c_9 \) are relatively complicated coefficients. For example, \( c_1 \) and \( c_2 \) take the form,

\[
c_1 = 1 - \frac{2i}{\omega} \left[ (\cos \theta)^2 \frac{\partial v_z}{\partial x} + (\sin \theta)^2 \frac{\partial v_y}{\partial y} + \frac{\sin 2\theta}{2} \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \right]
\]

(3)
The exact expressions for $c_3$ through $c_9$ are given in Appendix A. As can be seen, these coefficients depend on the mean velocity $v_i$, and its derivatives. Here, the velocities and their derivatives are expressed in Cartesian coordinates since the mean velocity is often better represented in that way. Also, these coefficients can be calculated easily before the propagation equation is solved.

If Eqn. (2) is divided by $c_1$, we obtain,

$$\frac{\partial^2}{\partial r^2} + k^2(1+\eta) + \sum_{i=3}^{9} \frac{c_i}{c_1} \frac{\partial}{\partial r} + \frac{c_2}{c_1} \frac{\partial}{\partial \theta} + \frac{c_4}{c_1} \frac{\partial^2}{\partial \theta^2} + \frac{c_5}{c_1} \frac{\partial^2}{\partial z^2} + \frac{c_6}{c_1} \frac{\partial^2}{\partial r \partial z} + \frac{c_7}{c_1} \frac{\partial^2}{\partial \theta \partial z} + \frac{c_8}{c_1} \frac{\partial^2}{\partial r \partial \theta} + \frac{c_9}{c_1} \frac{\partial}{\partial z} \biggr| p(r,\theta,z) = 0$$

where, $k^2(1+\eta) = k^2(1+\epsilon)/c_1$ and $\eta = (1+\epsilon - c_1)/c_1$. This is the 3D Helmholtz equation in cylindrical polar coordinates for an arbitrary moving inhomogeneous medium. The only difference between Eqns. (1) and (5) is the change of the coordinate system. Therefore, they are both based on the same three assumptions.

If there are no wind or density gradients in the medium, the expressions for the coefficients $c_1, c_2, ..., c_9$ can be easily reduced to the following forms,

$$c_1 = 1, c_2 = \frac{1}{r}, c_3 = 0, c_4 = \frac{1}{r^2}, c_5 = 1, c_6 = 0, c_7 = 0, c_8 = 0, c_9 = 0$$

Substituting these coefficients into Eqn. (5) gives the 3D Helmholtz equation for a uniform medium at rest,

$$\frac{\partial^2}{\partial r^2} + k^2 + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \biggr| p(r,\theta,z) = 0$$

Eqn. (7) has been used in a previous three-dimensional propagation model (FOR3D) to describe the effects of azimuthal variations in ocean acoustics by Lee et al., where the effect of the current velocity is added into the wavenumber $k$. Moreover, comparing Eqns. (7) and (5) shows that the wind not only changes the coefficients of the derivative terms that exist in the truly 3D homogeneous Helmholtz equation, but also introduces new cross derivative terms.

### B. Parabolic equation for a moving inhomogeneous medium in cylindrical coordinates

Eqn. (5) is the starting point for the derivation of a parabolic equation for sound propagation in a moving inhomogeneous medium in cylindrical coordinates. The approach combines the strategies used for the derivation of a homogeneous parabolic equation by Lee et al. and Ostashev et al.’s approach for the derivation of a 3D parabolic equation in Cartesian coordinates. First, we assume that the zeroth-order Hankel function of the second kind, $H_0^{(2)}(kr)$, is still a part of the solution of the new 3D Helmholtz equation, as in the 2D PE approach, and that it can be approximated by $\exp(ikr)/\sqrt{r}$. Setting $p = q/\sqrt{r}$ gives,

$$\frac{\partial^2}{\partial r^2} + k^2(1+\eta) + \sum_{i=3}^{9} \frac{c_i}{c_1} \frac{\partial}{\partial r} + \frac{c_2}{c_1} \frac{1}{r} \frac{\partial}{\partial r} + \frac{c_3}{c_1} \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{c_4}{c_1} \frac{\partial^2}{\partial \theta^2} + \frac{c_5}{c_1} \frac{\partial^2}{\partial z^2} + \frac{c_6}{c_1} \frac{\partial^2}{\partial r \partial z} + \frac{c_7}{c_1} \frac{\partial^2}{\partial \theta \partial z} + \frac{c_8}{c_1} \frac{\partial^2}{\partial r \partial \theta} + \frac{c_9}{c_1} \frac{\partial}{\partial z} \biggr| q = 0$$

where, $\eta = \eta - k^{-2}[3/(4r^2) - 1/(2r)c_2/c_1]$. To simplify the above expression, we can write

$$d_1 = \frac{c_2}{c_1} - \frac{1}{r}, d_2 = \frac{c_3}{c_1} - \frac{1}{2r}c_1, d_3 = \frac{c_4}{c_1}, d_4 = \frac{c_5}{c_1}, d_5 = \frac{c_6}{c_1}, d_6 = \frac{c_7}{c_1}, d_7 = \frac{c_8}{c_1}, d_8 = \frac{c_9}{c_1} - \frac{1}{2r}c_1$$

Furthermore, Eqn. (8) can be expressed in the following operator notation,

$$\left(\frac{\partial^2}{\partial r^2} + k^2 Q^2\right) q = 0$$

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Here, $Q$ is an operator including the derivatives in the $\theta$ and $z$ directions, which can be written as

$$Q = (1 + L)^{1/2}, \quad L = F + G \frac{\partial}{\partial r}$$

(11)

where,

$$F = \eta' + \frac{d_2}{k^2} \frac{\partial}{\partial \theta} + \frac{d_3}{k^2} \frac{\partial^2}{\partial \theta^2} + \frac{d_4}{k^2} \frac{\partial^2}{\partial z^2} + \frac{d_6}{k^2} \frac{\partial^2}{\partial \theta \partial z} + \frac{d_8}{k^2} \frac{\partial}{\partial z}$$

(12)

and,

$$G = \frac{d_1}{k^2} + \frac{d_5}{k^2} \frac{\partial}{\partial z} + \frac{d_7}{k^2} \frac{\partial}{\partial \theta}$$

(13)

The operator $Q$ can be represented by a Padé (1,1) approximation. This has been commonly used in previous derivations of the parabolic equation. That is,

$$Q = (1 + L)^{1/2} \sim p_1 + p_2 L q_1 + q_2 (F + G \frac{\partial}{\partial r})$$

(14)

where,

$$p_1 = 1, p_2 = \frac{3}{4}, q_1 = 1, q_2 = \frac{1}{4}$$

(15)

Eqn. (10) can now be split into an incoming and an outgoing wave. Here, only the outgoing wave is considered. Then substituting Eqn. (14) into the outgoing wave component yields,

$$\frac{\partial q}{\partial r} = ik \frac{p_1 + p_2 (F + G \frac{\partial}{\partial r})}{q_1 + q_2 (F + G \frac{\partial}{\partial r})} q$$

(16)

or,

$$[q_1 + q_2 F - ik p_2 G] \frac{\partial q}{\partial r} = ik (p_1 + p_2 F) q - q_2 G \frac{\partial^2 q}{\partial r^2}$$

(17)

The right hand side of this equation contains a term involving $\frac{\partial^2 q}{\partial r^2}$. To obtain a parabolic equation in the range direction, this term has to be eliminated. In order to do that, Eqn. (5) is written as,

$$\left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} + k^2 + O (\frac{|v/c_0|}{\theta}, \frac{|\tau/c_0|}{\theta}, \frac{|\rho/\rho_0|}{\theta}) \right] p = 0$$

(18)

Here, the assumption is used that the wind velocity and sound speed deviations are small compared to the sound speed, and the density fluctuation is small compared to the ambient density. This assumption is reasonable for most atmospheric propagation situations. Introducing $p = q/\sqrt{r}$ into Eqn. (18) gives,

$$\frac{\partial^2 q}{\partial r^2} = - \left[ k^2 + \frac{1}{4r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right] q$$

(19)

Then, the substitution of Eqn. (19) into Eqn. (17) yields

$$[q_1 + q_2 F - ik p_2 G] \frac{\partial q}{\partial r} = ik (p_1 + p_2 F) q + q_2 G \left( k^2 + \frac{1}{4r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) q$$

(20)

Furthermore, with $q = \psi \exp(ikr)$, we obtain,

$$[q_1 + q_2 F - ik p_2 G] \frac{\partial \psi}{\partial r} = ik (p_1 + p_2 F - q_1 - q_2 F + ik p_2 G) \psi + q_2 G \left( k^2 + \frac{1}{4r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) \psi$$

(21)

Eqn. (21) together with Eqns. (12) and (13) represent a 3D parabolic equation in cylindrical coordinates for an arbitrary moving inhomogeneous medium.
III. Numerical Approach

A. The Crank-Nicholson finite difference approach

To solve the resulting parabolic equation numerically, a Crank-Nicholson PE (CNPE) method is used in the present study for the same stability reason as for the 2D PE approach.\textsuperscript{14} In order to facilitate the discretization, Eqn. (21) is first written in operator form,

\[ A \frac{\partial \psi}{\partial r} = B \psi \]  \hspace{1cm} (22)

Here, \( A \) and \( B \) are operators in the \( z \) and \( \theta \) directions. Applying the Crank-Nicholson finite difference algorithm in the \( r \) direction yields,

\[ \left( A - \frac{\Delta r}{2} B \right) \psi(r + \Delta r) = \left( A + \frac{\Delta r}{2} B \right) \psi(r) \]  \hspace{1cm} (23)

To obtain the operators \( A \) and \( B \), second order accurate central difference schemes are used:

\[ \frac{\partial \psi}{\partial z} = \frac{\psi(z + \Delta z, \theta) - \psi(z - \Delta z, \theta)}{2\Delta z} \]  \hspace{1cm} (24)

\[ \frac{\partial^2 \psi}{\partial z^2} = \frac{\psi(z + \Delta z, \theta) - 2\psi(z, \theta) + \psi(z - \Delta z, \theta)}{\Delta z^2} \]  \hspace{1cm} (25)

\[ \frac{\partial^3 \psi}{\partial z^3} = \frac{\psi(z + 2\Delta z, \theta) - 2\psi(z + \Delta z, \theta) + 2\psi(z - \Delta z, \theta) - \psi(z - 2\Delta z, \theta)}{2\Delta z^3} \]  \hspace{1cm} (26)

Similar operations are applied in the \( \theta \) direction and for the mixed derivative terms. After a lengthy but straightforward process, a linear system of equations can be obtained. This is the finite difference form of Eqn. (23). Since third order derivatives are included in the operators \( A \) and \( B \), five-point finite difference stencils are used in both the \( z \) and \( \theta \) directions. In addition, mixed derivatives occur. Therefore, in all, there are 13 unknown variables in each of the linear system of equations. These unknown variables are \( \psi(z, \theta - 2\Delta \theta), \psi(z - \Delta z, \theta - \Delta \theta), \psi(z, \theta - \Delta \theta), \psi(z + \Delta z, \theta, \theta - \Delta \theta), \psi(z - 2\Delta z, \theta), \psi(z - \Delta z, \theta), \psi(z, \theta), \psi(z + \Delta z, \theta), \psi(z + 2\Delta z, \theta), \psi(z - \Delta z, \theta + \Delta \theta), \psi(z, \theta + \Delta \theta), \psi(z + \Delta z, \theta + \Delta \theta), \) and \( \psi(z, \theta + 2\Delta \theta) \) as shown in Fig. 1. The coefficients of these variables depend on the sound speed, the density of the air, the wind speed, their gradients, and the wavenumber.

![Figure 1. The discretization strategy.](image)

One issue has to be stressed in solving the resulting linear system. To march in the range direction, it is necessary to solve a linear system of \( N \) equations with \( N \) unknown variables at each range \( r \) step. Here, the \( N \) unknown variables correspond to the acoustic pressure at each mesh point of the \((z, \theta)\) plane. It is well known
that solving this system of equations requires $N^3$ calculations if a direct method, such as LU decomposition algorithm, is adopted. For example, if a 3D case with 2000 steps in the height direction and 181 steps in the azimuthal direction is considered, there will be $2000 \times 181 = 362,000$ points in one cylindrical surface of the computational domain. That is, 362,000 equations corresponding to 362,000 unknown variables has to be solved. To solve a linear system of 362,000 equations requires $(362,000)^3$ operations if the LU algorithm is used. For a 3.06 GHz Intel Xeon CPU, this would require $(362,000)^3/(3.06 \times 10^9) \approx 1.58 \times 10^7$s.

That is, about 183 days are required for one step of range marching on such a CPU. This is definitely not practical for a long range propagation problem and it is why our preliminary calculations were limited to a very small computational domain. Obviously, a more efficient solver is needed. Here, an iterative solver, the Generalized Minimum RESidual (GMRES) method is adopted. The cost of the iterations grows with number of iterative, thus the method is restarted after a number of iterations. Generally, an iterative algorithm requires less than $N^2$ calculations for a linear system with $N$ equations. For the above computational domain setup, only 43.7 second is needed for one calculation step with the 3.06 GHz CPU. Since the GMRES method reduces the computational effort significantly, marching by the 3D PE method can now be performed in a reasonable time frame.

B. The initial field

In previous research for 2D axisymmetric PE calculations, a Gaussian starting field has often been used.\(^2\)

\[
p(r = 0, z) = \sqrt{ik} \exp \left[ \frac{-k^2(z - z_s)^2}{2} \right]
\]

Here, the point source is located at $(0, 0, z_s)$. This form avoids the singular point in the analytical solution if it is calculated at $r = 0$. However, in the 3D PE model the coefficient of $\partial \psi / \partial \theta$ in Eqn. (20) has a $1/r$ term, which could generate a potential problem for the matrix equation as $r \to 0$ since a second order central difference scheme is used. The problem is that the coefficients of $\psi(z, \theta - \Delta \theta)$ and $\psi(z, \theta + \Delta \theta)$ could be much larger than the coefficient of $\psi(z, \theta)$, which is the diagonal term of the coefficient matrix. Consequently, the resulting matrix is stiff with ensuing numerical convergence and accuracy issues. Fortunately, this problem can be avoided if the initial field is set at a finite distance away from the acoustic source, where the $1/r$ and $1/r^2$ terms are no longer large and the resulting coefficient matrix is diagonally dominant. Furthermore, since there is no singularity in this approach, the analytical solution can be used as the initial field.

The acoustic field of a point source in an unbounded, uniform atmosphere is given by,

\[
p_{\text{free}}(r, z) = \frac{\exp(ikR)}{R}
\]

where $R = \sqrt{r^2 + (z - z_s)^2}$ is the radial distance from the source located at $(0, 0, z_s)$ to the observer. Including the reflection effect due to a ground surface, the acoustic pressure at $(r, z)$ can be written as,

\[
p(r, z) = \frac{\exp(ikR_1)}{R_1} + C_r \frac{\exp(ikR_2)}{R_2}
\]

Here, $C_r$ is the reflection coefficient, $R_1 = \sqrt{r^2 + (z - z_s)^2}$ and $R_2 = \sqrt{r^2 + (z + z_s)^2}$. This analytical solution is only applicable to a homogeneous atmosphere, which could be used for the initial field in the no wind example case.

The simplest moving medium sound propagation problem is that where the wind velocity is uniform. If the wind velocity is in the $x$ direction, the problem can be described by,

\[
(-ik + M \frac{\partial}{\partial x})^2 g(x|x_s) - \nabla^2 g(x|x_s) = 2\pi \delta(x - x_s)
\]

where $M$ is the Mach number in the $x$-direction, $k$ is the wavenumber, $x = (x, y, z)$ is the observer location, $x_s = (x_s, y_s, z_s)$ is the source location, and $g(x|x_s)$ is the Green’s function. The solution of Eqn. 30 is given by,

\[
g(x|x_s) = \frac{\gamma^2}{R} \exp \left[ ik(R - M\gamma^2(x - x_s)) \right]
\]

Here, $\gamma = 1/\sqrt{1 - M^2}$, and $R = \gamma \sqrt{\gamma^2(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2}$. Eqn. (31) gives the acoustic field of a point source for a uniform wind in $x$ direction. Similar to the homogeneous atmosphere case, the analytical
solution for a point source in a uniform wind above a flat reflecting ground surface can be obtained by adding an image source at \((0, 0, -z_s)\). This solution will also be used as an initial field later and for a validation case in next section. It has to be pointed out that the uniform wind case is just used for validation of the 3D PE method, since a uniform wind does not exist in practice due to the friction of the ground surface.

C. Boundary conditions

Besides the initial field, two kinds of boundary conditions need to be considered for a 3D computational domain: one is in the azimuthal direction and the other is in the vertical direction.

If a full circle or 360° in the azimuthal direction is considered for the computational domain, periodicity can be used for the azimuthal boundary condition. For example, the pressure field at \(-90°\) is equal to the pressure field at \(270°\). On the other hand, if the problem is symmetrical about the \(y\)-direction (along the \(-90°\) to \(90°\) line in cylindrical coordinates as shown in Fig. 2, the 360° computational domain can be reduced to a 180°, which consequently reduces the computational demands.

![Figure 2. The computational domain for the 3D PE calculation.](image)

In setting the vertical boundary, there is a difficulty due to the second order finite difference used for the third order differential terms in Eqn. 21. Consequently, the discretized PE involves \(\psi(z - 2\Delta z, \theta)\) and \(\psi(z + 2\Delta z, \theta)\) terms at the lower and upper boundaries, respectively. That is, two more mesh points are located outside of the vertical boundaries of the computational domain. This problem can be avoided if biased second order finite difference stencils are adopted for the third order derivative terms. They can be written as,

\[
\frac{\partial^3 \psi}{\partial z^3} \bigg|_{\text{lower boundary}} = \frac{2\psi(z + 2\Delta z, \theta) - 6\psi(z + \Delta z, \theta) + 6\psi(z, \theta) - 2\psi(z - \Delta z, \theta)}{3\Delta z^3}
\]

\[
\frac{\partial^3 \psi}{\partial z^3} \bigg|_{\text{upper boundary}} = \frac{2\psi(z + \Delta z, \theta) - 6\psi(z, \theta) + 6\psi(z - \Delta z, \theta) - 2\psi(z - 2\Delta z, \theta)}{3\Delta z^3}
\]

Now, only one point beyond the upper and lower boundaries is needed for the discretization operations. The one-point-outside boundary problem can be handled by using a “locally reacting” boundary condition as in previous 2D PE simulations.\(^{18}\)

IV. Results and Discussion

The 3D parabolic equation, Eqn. (21) together with Eqns. (12) and (13), are used to simulate several example cases in this section. Validation of the 3D PE model is conducted by comparing numerical and analytic results for a no wind and a uniform wind case. Then a logarithmic wind profile case is simulated to investigate the wind influences. At the same time, the 3D PE results are compared to the 2D PE results.
that use an effective sound speed approximation for the wind velocity. Finally, a case is considered in which the wind magnitude and direction change with height.

Unless otherwise stated, these examples use a unit monopole source above a flat rigid ground surface with the observer located at 1.7 m above the ground. The frequency of the acoustic source is 20 Hz, the sound speed is 340 m/s, and there are no medium density and sound speed variations throughout the computational domain.

A. Homogeneous atmosphere with no wind

In order to validate the 3D PE model, the first example case conducted is sound propagation in a homogeneous atmosphere with no wind. The acoustic source is located 3.4 m above the ground surface. Since a homogeneous atmosphere is assumed, the problem is strictly axisymmetric and there is no \( \theta \) dependence. This problem can be described by Eqn. (7) and it has an analytical solution given by Eqn. (29). To solve this case by the 3D numerical model, both the range and height steps are chosen to be one tenth of the wavelength. Although the problem is azimuthally independent, the azimuthal direction is still included in the simulation in order to validate the new 3D PE approach. The azimuthal range is set from \(-90^\circ\) to \(90^\circ\) and an azimuthal step size of \(15^\circ\) is used for the calculation. In general, \(90^\circ\) is set as the downwind direction, and \(-90^\circ\) is the upwind direction. At \(0^\circ\) the wind is perpendicular to the sound propagation direction and this is called the cross wind direction as in Fig. 2. Though, in this first example, there is no wind.

Figs. 3(a) and 3(b) compare the resulting real and imaginary parts of the acoustic pressure respectively with the analytical results. It can be seen that the 3D PE results match the analytical solutions very well. The resulting Sound Pressure Level (SPL) also matches the analytical solution very well as shown in Fig. 3(c). Furthermore, since there is no azimuthal dependence, the 3D PE results should be independent on the azimuthal direction. This is illustrated in Fig. 4, where the five curves in different azimuthal directions are all in agreement.

Figure 3. Comparison of the acoustic pressure from the analytical solution and the 3D PE numerical solution for the no wind case. The observer is located at \( z = 1.7 \text{m} \) and \( \theta = 90^\circ \). (a) Real part of the acoustic pressure, (b) imaginary part of the acoustic pressure, and (c) Sound Pressure Level (SPL).
B. Homogeneous atmosphere with an uniform wind

The second validation case is sound propagation in a homogeneous atmosphere with a uniform wind. The acoustic source is located at 3.4 m above the ground surface. The wind velocity is set in a direction from $-90^\circ$ to $90^\circ$ with an magnitude of $M = 0.2$. Clearly, this is not a realistic wind speed, but such a high value of $M$ was chosen to exaggerate the effect of the wind. An analytical solution is available as given by Eqn. (31). Moreover, with this equation, the initial acoustic field for the 3D PE simulation can be calculated on a cylindrical surface with a radius of 17 m, centered at the acoustic source. Since the problem is symmetrical, the semi-cylindrical computational domain is used with an azimuthal symmetry boundary condition.

Figs. 5(a) and 5(b) compare the numerical and analytical results for the real and imaginary parts of the acoustic pressure, respectively, in the cross wind direction, $\theta = 0^\circ$. Fig. 5(c) shows a comparison of the Sound Pressure Level (SPL). In each case the numerical results are in good agreement with the analytical results. Fig. 6 compares the real part of the acoustic pressure for the analytical and numerical results in the upwind and downwind directions. Again, the numerical and analytical results are in good agreement. The figures also show that the wavelength of the acoustic pressure is elongated in the downwind direction, and the wavelength is shortened in the upwind direction.

One issue that has to be addressed is the mesh resolution for the above calculation, where the range and height step sizes are chosen to be one tenth of the wavelength. In the azimuthal direction, a very small step size, $0.5^\circ$, is used. This means there are 361 azimuthal steps for the $180^\circ$ semi-cylindrical computational domain; that is, $m_\theta = 361$, where $m_\theta$ is the total number of steps in the azimuthal direction. Notice, that in this case the relative wind velocity varies with azimuthal angle, and thus the azimuthal step could affect the accuracy of the numerical simulation. Fig. (7) shows the influence of azimuthal mesh resolution, $m_\theta$, on the numerical errors for the real part of the acoustic pressure. The figure shows that the finer meshes give better accuracy for the numerical simulations. A similar conclusion can be drawn from the imaginary part of the acoustic pressure. As can be seen, the $m_\theta = 91$ results are very close to the $m_\theta = 361$ results. Thus, $m_\theta = 91$ results with $2^\circ$ resolution in azimuthal direction can be treated as the converged results are used for the subsequent simulations.
Figure 5. Comparison of the analytical solution and the 3D PE numerical solution for the uniform wind case. The observer is located at $z = 1.7$ m and $\theta = 0^\circ$. (a) Real part of the acoustic pressure, (b) imaginary part of the acoustic pressure, and (c) Sound Pressure Level (SPL).

Figure 6. Comparison of the real part of the acoustic pressure from the analytical solution and the 3D PE numerical solution for the uniform wind case, at different downwind ($\theta = 90^\circ$) and upwind locations ($\theta = -90^\circ$).
Figure 7. Comparison of numerical errors for the real part of the acoustic pressure with different azimuthal resolution \((m_\theta)\) for the uniform wind case.

C. Homogeneous atmosphere with a logarithmic wind

Due to the friction of the ground surface, the wind profile is generally considered to be logarithmic in the boundary layer of the atmosphere, and can be described by

\[
v = A \ln(z + 1.0)
\]  \hspace{1cm} (34)

Here \(A\) is a constant, and \(z\) is the height. If the wind direction is from \(-90^\circ\) and \(90^\circ\) and only the wind magnitude varies with height, the problem is still symmetrical. Thus a semi-cylindrical computational domain can be used. Again there are no sound speed or density gradients in this example case. \(A = 9.424\) is used, which gives \(M \leq 0.2\) or \(v = 68\) m/s at the largest height (1360 m) of the computational domain as shown in Fig. 8.

Figure 8. The velocity profile for the logarithmic wind with \(A = 9.424\).

There is no analytical solution for this problem. However, to set the initial field, the analytical solution, Eqn. (31) for the uniform wind case is still adopted for the 3D PE calculation. The initial cylindrical surface is still located 17 m away from the acoustic source. The problem can be thought of as such an initial acoustic field being propagated in an atmosphere with a logarithmic wind profile. Here, two example cases are modeled with two different source locations: \(z_s = 3.4\) m and \(z_s = 68\) m.

Fig. 9(a) shows the resulting SPL for the \(z_s = 3.4\) m case as a function of range, at an observer height of \(z = 1.7\) m and five different azimuthal locations. Again, \(\theta = -90^\circ\) is the upwind direction, \(\theta = 0^\circ\) is the
cross wind direction, and $\theta = 90^\circ$ is the downwind direction. Clearly, the SPL profiles are very different from each other in the different observer directions. In the downwind direction the SPL has the largest values due to downward refraction, where peaks and valleys alternate in the SPL profile because of the multiple ground reflections. In the cross wind directions, there is no acoustic refraction in the vertical direction, and the SPL is only slightly affected by the cross wind. Multiple ground reflections do not exist either, thus the SPL profile is smooth. In the upwind direction the SPL decreases with range due to upward refraction, and a shadow zone is formed as the range reaches about 600 m. (In an upward refracting atmosphere, a region exists where no sound rays arrive, this region is called the shadow region or shadow zone. In this paper, the shadow zone is taken to be the locations where the SPL is less than zero.) At $\theta = -30^\circ$ weaker upward refraction occurs, and a shadow zone forms at a longer range compared to the $\theta = -90^\circ$ direction. Fig. 9(b) shows the same comparison as in Fig. 9(a) only with the range axis plotted on a logarithmic scale. It should be stressed that the influence of atmospheric absorption is not considered in these simulations, but it could be added into the PE model by adding an imaginary part to the wave number.

Figure 9. Comparison of SPL for different observer azimuthal angles for the logarithmic wind case from the 3D PE calculation. Acoustic source is located at 3.4 m. (a) Range axis in linear scale, and (b) range axis in logarithmic scale.

Fig. 10 shows the SPL profiles for the $z_s = 68$ m case. Different from the $z_s = 3.4$ m case, a shadow zone begins at a longer range, about $r = 800$ m, in the upwind direction. This is because of the larger vertical distance between the acoustic source and observer. A similar trend can be found for an observer at $\theta = -30^\circ$. Meanwhile, in the downwind direction, the SPL is less than in the $z_s = 3.4$ m case. This is not only because the vertical distance between the source and observer increases, but also because fewer ground reflections occur. Consequently, fewer peaks and valleys alternate for the $z_s = 68$ m case than for the $z_s = 3.4$ m case. A similar influence can be also found for the observers at $\theta = 30^\circ$. At the observer location $\theta = 0^\circ$ the SPL for the $z_s = 68$ m case is less than the $z_s = 3.4$ m case at smaller ranges ($r < 100$ m), because the vertical distance between acoustic source and observer is larger. Yet with an increase in range, the influence of the vertical distance between source and observer reduces, and thus the SPL of the two cases ($z_s = 68$ m case and $z_s = 3.4$ m case) are similar at long ranges.

To identify the effectiveness of the use of the 2D PE model in the case of a 3D wind, Fig. 11 shows contours of the SPL difference between the 2D and 3D PE results at an observer height of $z = 1.7$ m for the $z_s = 3.4$ m case. In the downwind locations, noticeable differences can be found, where the maximum difference is about 3.8 dB. In the upwind direction, much larger differences can be found. However, in the upwind direction the SPL is less than 0 dB at most locations, so these differences can be accounted for by small numerical errors. For a clearer comparison, Fig. 12 shows the predicted SPL in different azimuthal
Figure 10. Comparison of SPL for different azimuthal angles for the logarithmic wind case from the 3D PE calculation. Acoustic source is located at 68 m. (a) Range axis in linear scale, and (b) range axis in logarithmic scale.

directions for the 3D PE and the 2D PE results. Fig. 12(a) shows an upwind and downwind comparison. As can be seen, the differences are not more than 3 dB in either direction. However, noticeable SPL differences can be found in the cross wind direction ($\theta = 0^\circ$) in Fig. 12(b). These can be as large as 5 dB. Since there is no wind in the direction parallel to sound propagation, the SPL differences can be accounted for by the neglect of the cross wind in the 2D PE method. Fig. 13 shows contours of the SPL difference between the 2D and 3D PE results at $z = 1.7$ m for the $z_s = 68$ m case. Again, the SPL differences in the upwind direction can be accounted for by small numerical errors. In the downwind direction, differences larger than 14 dB can be found in many locations. Fig. 14 compares the predicted SPL at different azimuthal locations from the 3D PE and 2D PE results. Significant differences can be seen in the downwind direction, shown in Fig. 14(a). The maximum difference is more than 20 dB. For observers at $\theta = 0^\circ$, insignificant differences can be identified between 2D and 3D PE results. The maximum SPL difference is about 3 dB.

Fig. 14 shows that for the $z_s = 68$ m case, the SPL difference between the 3D and 2D PE results in the downwind direction increases significantly with range, and yet the differences in the crosswind direction are small. Pierce$^{10}$ gives an equation to describe the curvature of a refracted sound ray due to a wind gradient in the vertical direction on the sound propagation surface, without considering the lateral refraction caused by crosswinds,

$$ r_c = \frac{c}{(dc/dz) \sin \theta + dv_x/dz} $$

Here $v_x$ is the wind component in the sound propagation direction, and $\theta$ is the incident angle between the sound ray and the vertical direction (height). If a sound ray propagates in a nearly horizontal direction, $\sin \theta$ is approximately unity, and the effective sound speed is an appropriate approximation to the effects of the wind. However, if $\theta$ is less than $30^\circ$, the influence of a wind-speed gradient is substantially greater than that of a sound-speed gradient of the same magnitude. Under this condition, the effective sound speed is not an appropriate approximation anymore. For the $z_s = 68$ m case, the incident angle is larger than $30^\circ$, and that is still true even at long ranges due to the multiple reflections from the ground. Since the 2D PE still uses the effective sound speed approximation for this case, more significant differences are identified. For the $z_s = 3.4$ m case the incident angle is less than $30^\circ$, and thus no large difference can be found in the downwind direction. In the crosswind directions, the wind velocity is perpendicular to the sound propagation direction, and thus no effective sound speed approximation is necessary in the 2D PE. The SPL difference can only be due to the cross wind. At the observer height $z = 1.7$, the wind velocity is approximately
Figure 11. Contour of the SPL difference between the 2D and 3D PE results at an observer height of $z = 1.7$ m, for a monopole source located at $z_s = 3.4$ m and a logarithmic wind case.

Figure 12. SPL comparison of numerical results at $z = 1.7$ m and different azimuthal locations, for calculations with the 3DPE and the 2DPE with effective sound speed. A monopole source is located at $z_s = 3.4$ m for a logarithmic wind profile.
Figure 13. Contours of the SPL difference between the 2D and 3D PE results at an observer height $z = 1.7$ m, for a monopole source located at $z = 68$ m and a logarithmic wind profile.

Figure 14. SPL comparison of numerical results at $z = 1.7$ m and different azimuthal locations, for calculations with the 3DPE and the 2DPE with effective sound speed. A monopole source is located at $z_s = 68$ m for a logarithmic wind profile.
8.3 m/s ($M = 0.02$) and is relatively small. Thus, there is no significant SPL difference due to the lateral refraction for both cases. For observer locations $\theta = -90^\circ$ and $\theta = -30^\circ$, since the influence of upward refraction is dominant and a shadow zone forms in a very small distance, there is not enough distance to show SPL differences between the 2D and 3D PE results. At observer locations of $\theta = 30^\circ$, slightly larger SPL differences than that at $\theta = 0^\circ$ can be found. However, where the wind velocity component in the sound propagation direction is small, no significant SPL differences can be found.

D. Homogeneous atmosphere with a rotating wind

In all the cases considered so far, the wind velocity has had a simple profile. In this subsection, a case with an assumed more complicated three-dimensional wind profile is considered. Here, not only the magnitude of the wind velocity changes with height as in the logarithmic wind profile, but also the direction of the wind changes with height. The wind direction profile is shown in Fig. 15, and this case is called the rotating wind case. The resulting velocity components, $v_x$ (along the $0^\circ$ and $180^\circ$ axis) and $v_y$ (along the $-90^\circ$ and $90^\circ$ axises), are shown in Fig. 16. For this case, a more significant wind shear is considered with increased wind velocity varying with height. Notice that the problem is not symmetrical anymore. Thus, a full cylindrical computational domain has to be used with a periodic boundary condition. Again the initial field is calculated at a cylindrical surface 17 m from the acoustic source. Two cases are simulated: one has the source at a height of 3.4 m and the other at a source height of 68 m.

![Figure 15. Wind velocity direction for the rotating wind case.](image)

Fig. 17 shows the variations of the SPL with range for three different observer directions for a source located at $z_s = 3.4m$. Fig. 17(a) shows the range from 3000 m to 6000 m and Fig. 17(b) shows an enlarged view for the range from 4000 m to 5000 m. The acoustic pressure is significantly different at the different observer locations. In this figure, 2D PE results, using the effective sound speed, are also included for comparison. However, as discussed before, the 2D simulation cannot model the influence of a cross wind. Here the vertical distance between source and observer is small, and the effective sound speed can be considered as a good approximation to the wind velocity component parallel to the sound propagation direction for the 2D calculation. Thus the difference between the 3D and 2D results are mainly due to the cross wind and wind shear. As can be seen, there are noticeable differences between the 3D and 2D results. The maximum difference is found to be more than 20 dB. Fig. 18 shows contours of the SPL difference between the 2D and 3D PE results at $z = 1.7$ m for the $z_s = 3.4$ m case, and again SPL differences of more than 14 dB can be found at long ranges.

Fig. 19 shows a similar comparison for a source height $z_s = 68m$ case. However, due to the larger height difference between the source and observer, the effective sound speed should not be expected to be a good
Figure 16. Wind velocity amplitude of the rotating wind.

Figure 17. SPL comparison of numerical results using 2DEP with an effective speed of sound approximation and 3DPE for a monopole source located at $z_s = 3.4$ m and different azimuthal locations for the rotating wind case. (a) Range 3000-6000 m and, (b) range 4000-5000 m.
Figure 18. Contours of SPL difference between the 2D and 3D PE results at an observer height $z = 1.7$ m for a monopole source located at $z_s = 3.4$ m for the rotating wind case.

approximation to the wind velocity for the 2D PE calculation. More significant differences between the 3D and 2D calculations can be found. Fig. 20 shows contours of the SPL differences between 2D and 3D PE results at an observer height $z = 1.7$ m for the $z_s = 68$ m case. Much more significant SPL differences can be found than those in the $z_s = 3.4$ m case. Based on these comparisons, it can be concluded that, for a more complicated wind case with both direction and magnitude changes with height, the 2D PE does not predict the influence of wind on sound propagation accurately.

Figure 19. SPL comparison of the numerical results using 2DPE with an effective speed of sound approximation and 3DPE for a monopole source located at $z_s = 68$ m at different azimuthal locations for the rotating wind case. (a) Range 3000-6000 m and, (b) range 4000-5000 m.
V. Conclusions

In this paper, a 3D Helmholtz equation in cylindrical coordinates is derived for an arbitrary moving inhomogeneous medium. Based on this equation, a novel 3D parabolic equation in cylindrical coordinates is developed. The parabolic equation is then discretized by the implicit Crank-Nicholson finite difference method with second order accuracy, and the resulting system of linear equations is solved using the GMRES method with an appropriate initial field and boundary conditions.

To validate the 3DPE model, several example cases have been calculated. The results are compared to the analytical solutions for the no wind and uniform wind cases, where the numerical and analytical results match very well. Then the 3DPE model is applied to an investigation of propagation with a logarithmic wind profile. The numerical results show the atmospheric refraction due to the vertical wind velocity gradients clearly. Then, the 3DPE results are treated as accurate solutions to identify the SPL differences between the 3DPE and 2DPE results with an effective sound speed approximation for wind velocity. If the acoustic source height is close to that of the observer, the 2DPE errors are small. However, as the vertical height difference increases, significant differences can be identified. In the downwind directions the maximum SPL differences can be more than 20 dB. It can be concluded that the traditional 2D PE results are acceptable if the acoustic source and the observer have similar heights (when there is no cross wind). However, as the height difference between the source and the observer increases, the use of an effective sound speed in the 2D PE is not effective anymore and 3D PE method should be used. This conclusion agrees with the analysis by Pierce.\textsuperscript{10} In addition, the 3D PE has been used to simulate a 3D rotating wind case, where the wind is found to have a significant influence on sound propagation. Even if the height difference between the source and observer are small, the difference between 2D PE and 3D PE results are very significant. So, it can be concluded that for a complicated wind case, a 3D PE method should be used to accurately describe the wind velocity’s influence on sound propagation. Finally, the wind magnitude in the cases given here has been chosen to be unrealistically high - to emphasize the effects of the wind. In reality, the differences would be smaller. In such cases the computationally less expensive 2DPE would provide a reasonable approximation in the overall SPL trends. However, for locally more accurate results, the 3DPE formulation should be used.

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VI. Appendix A

Expressions for the coefficients $c_3, ..., c_9$ in Eqn. (2)

\[
\begin{align*}
    c_3 &= \frac{1}{\rho^2} \frac{\partial \rho}{\partial \theta} - \frac{2i}{\omega} \left[ \frac{\sin 2\theta}{r^2} \frac{\partial v_x}{\partial x} - \frac{\sin 2\theta}{r^2} \frac{\partial v_y}{\partial y} - \frac{\cos 2\theta}{r^2} \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \right] \\
    c_4 &= \frac{1}{r^2} - \frac{2i}{\omega} \left[ \frac{(\sin \theta)^2}{r^2} \frac{\partial v_z}{\partial x} + \frac{(cos \theta)^2}{r^2} \frac{\partial v_y}{\partial y} - \frac{\sin 2\theta}{2r^2} \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \right] \\
    c_5 &= 1 - \frac{2i}{\omega} \frac{\partial v_z}{\partial z} \\
    c_6 &= -\frac{2i}{\omega} \left[ \cos \theta \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} \right) + \sin \theta \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_z}{\partial y} \right) \right] \\
    c_7 &= -\frac{2i}{\omega} \left[ -\frac{\sin \theta}{r} \left( \frac{\partial v_z}{\partial x} + \frac{\partial v_z}{\partial z} \right) + \cos \theta \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_y}{\partial y} \right) \right] \\
    c_8 &= -\frac{2i}{\omega} \left[ -\frac{\sin 2\theta}{r} \frac{\partial v_x}{\partial x} + \frac{\sin 2\theta}{r} \frac{\partial v_y}{\partial y} + \frac{\cos 2\theta}{r} \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \right] \\
    c_9 &= -\frac{1}{\rho} \frac{\partial \rho}{\partial z} + \frac{2ik}{c_o} v_z
\end{align*}
\]

References


