Math 230, Fall 2006

Solutions to Midterm Exam 1

Problem 1.

1. Find an equation of the sphere with center \( P = (1, 1, 2) \) and passing through \( Q = (2, 1, 1) \).

2. Find the intersection points (if any) of this sphere and the \( z \)-axis.

3. Find the intersection points (if any) of this sphere and the \( x \)-axis.

Solutions.

1. \( \vec{PQ} = <1, 0, -1> \) and \( |\vec{PQ}| = \sqrt{2} \).

Hence, an equation of the sphere with center \( P = (1, 1, 2) \) and passing through \( Q = (2, 1, 1) \) is \( (x - 1)^2 + (y - 1)^2 + (z - 2)^2 = 2 \).

2. Setting \( x = 0 \) and \( y = 0 \), we get

\[
(-1)^2 + (-1)^2 + (z - 2)^2 = 2 \quad \text{or} \quad (z - 2)^2 = 0.
\]

The solution of this equation is \( z = 2 \). So, the intersection point of the sphere and the \( z \)-axis is \((0, 0, 2)\).

3. Similarly, setting \( y = z = 0 \), we get

\[
(x - 1)^2 + (-1)^2 + (-2)^2 = 2 \quad \text{or} \quad (x - 1)^2 = -3.
\]

There is no solution for this equation. Consequently, the sphere does not intersect the \( x \)-axis.
**Problem 2.** Find the volume of the parallelepiped with adjacent edges $\vec{PQ}$, $\vec{PR}$ and $\vec{PS}$ if $P = (1, 1, 2)$, $Q = (2, 3, 1)$, $R = (-1, 1, 5)$ and $S = (1, 8, -2)$.

**Solution.** The volume of this parallelepiped is given by

$$ V = |\vec{PQ} \cdot (\vec{PR} \times \vec{PS})|. $$

Since

$$ \vec{PQ} = <1, 2, -1> \quad \vec{PR} = <-2, 0, 3>, \quad \text{and} \quad \vec{PS} = <0, 7, -4> $$

one gets

$$ \vec{PR} \times \vec{PS} = -21i - 8j - 14k. $$

Therefore, $V = |-21 - 16 + 14| = |-23| = 23$

**Problem 3.** If the spherical coordinates of a point $P$ are $(\rho, \theta, \phi) = (\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4})$,

1. find its rectangular coordinates.
2. Find its cylindrical coordinates.

**Solutions.**

1. Since $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi$, one gets $(x, y, z) = (0, 1, 1)$.

2. But $r = \sqrt{x^2 + y^2} = 1$, hence $(r, \theta, z) = (1, \frac{\pi}{2}, 1)$.

**Problem 4.**

1. Find parametric equations of the line segment from $P = (1, -1, 2)$ to $Q = (3, 2, 1)$.

2. Find the intersection point (if any) of the $xy$-plane and the line through $P$ and $Q$.

**Solution.**

1. We have $\vec{PQ} = <2, 3, -1>$ hence parametric equations of the line segment from $P$ to $Q$ are:

$$ x = 1 + 2t, \quad y = -1 + 3t, \quad z = 2 - t, \quad \text{with} \quad 0 \leq t \leq 1. $$
2. Parametric equations of the line through $P$ and $Q$ are:

\[ x = 1 + 2t, \quad y = -1 + 3t, \quad z = 2 - t, \text{ where } t \text{ varies in } \mathbb{R}. \]

Setting $z = 0$, one gets $t = 2$. Replacing $t$ by this value, one obtains $(x, y, z) = (5, 5, 0)$ which is the intersection point of the $xy$-plane and the line through $P$ and $Q$.

**Problem 5.** A moving particle starts at the origin with initial velocity $v = 2i - j + k$. If its acceleration is $a(t) = 4t^2i + 12t^3j + 6tk$, find its position function.

**Solution.** One gets

\[ v(t) = r'(t) = \int_0^t a(u)du + v(0) = (2t^2 + 2)i + (4t^3 - 1)j + (3t^2 + 1)k. \]

It follows that

\[ r(t) = \int_0^t v(u)du + r(0) = \left( \frac{2}{3}t^3 + 2t \right)i + (t^4 - t)j + (t^3 + t)k. \]

**Problem 6.** Find symmetric equations of the tangent line to the curve given by the parametric equations $x = t^2 + 3t$, $y = t$, $z = t^4 + \sin t$ at the origin $O = (0, 0, 0)$.

**Solution.** One has

\[ r'(t) = < 2t + 3, 1, 4t^3 + \cos t >. \] Therefore, \[ r'(0) = < 3, 1, 1 >. \]

Symmetric equations of the tangent line to the curve at the origin are:

\[ \frac{x}{3} = y = z. \]

**Problem 7.** Given the vector function $r(t) = < \sin t - t \cos t, \cos t + t \sin t, -1 >$, find the length of its curve for $0 \leq t \leq 2$.  


Solution. We have \( r'(t) = \langle t \sin t, t \cos t, 0 \rangle \) and \( |r'(t)| = \sqrt{t^2} = t \) since \( 0 \leq t \leq 2 \). Therefore,
\[
L = \int_0^2 |r'(t)| \, dt = \left[ \frac{t^2}{2} \right]_0^2 = 2.
\]

Problem 8. Find an equation of the plane through \( P = (1, 3, 2) \), \( Q = (2, 4, 3) \), and \( R = (0, 3, 0) \).

Solution. We have
\[
\overrightarrow{PQ} = \langle 1, 1, 1 \rangle \quad \overrightarrow{PR} = \langle -1, 0, -2 \rangle \quad \overrightarrow{PQ} \times \overrightarrow{PR} = \langle -2, 1, 1 \rangle.
\]
An equation of the plane through \( P, Q \) and \( R \) is
\[
-2(x - 1) + (y - 3) + (z - 2) = 0 \quad \text{or} \quad -2x + y + z = 3.
\]

Problem 9. Suppose \( r(t) = \langle t, t^2, t^3 \rangle \) is the position function for a moving particle \( P \) at time \( t \). Find the normal component \( a_N \) of the acceleration at \( t = 1 \).

Solution. One gets
\[
r'(t) = \langle 1, t, t^2 \rangle, \quad r''(t) = \langle 0, 1, 2t \rangle \quad \text{and} \quad r'(1) \times r''(1) = \langle 1, -2, 1 \rangle.
\]
Hence,
\[
a_N = \frac{|r'(1) \times r''(1)|}{|r'(1)|} = \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}.
\]

Problem 10. Find the curvature of the curve \( C \) given by \( r(t) = \langle 4t, \sin t + 2 \cos t, 2 \sin t - \cos t \rangle \) at \( P = (0, 2, -1) \).

Solution: The point \( P \) corresponds to \( t = 0 \). Moreover,
\[
r'(t) = \langle 4, \cos t - 2 \sin t, 2 \cos t + \sin t \rangle \quad \text{and} \quad r'(0) = \langle 4, 1, 2 \rangle.
\]
\[
r''(t) = \langle 0, -\sin t - 2 \cos t, -2 \sin t + \cos t \rangle \quad \text{and} \quad r''(0) = \langle 0, -2, 1 \rangle.
\]
Therefore,
\[
\kappa = \frac{|r'(0) \times r''(0)|}{|r'(0)|^3} = \frac{|5i - 4j - 8k|}{21 \sqrt{21}} = \frac{\sqrt{105}}{21 \sqrt{21}} = \frac{\sqrt{5}}{21}.
\]